

Path regularity and explicit convergence rate for BSDE with truncated quadratic growth[☆]

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Abstract

We consider backward stochastic differential equations with drivers of quadratic growth (qgBSDE). We prove several statements concerning path regularity and stochastic smoothness of the solution processes of the qgBSDE, in particular we prove an extension of Zhang's path regularity theorem to the quadratic growth setting. We give explicit convergence rates for the difference between the solution of a qgBSDE and its truncation, filling an important gap in numerics for qgBSDE. We give an alternative proof of second order Malliavin differentiability for BSDE with drivers that are Lipschitz continuous (and differentiable), and then derive an analogous result for qgBSDE.

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1. Introduction

Backward Stochastic differential equations (BSDE) have been receiving much attention during the past 15 years, due to their central significance in optimization problems for instance

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in stochastic finance, and more generally in stochastic control theory. A particularly important class, BSDE with drivers of quadratic growth (qgBSDE) introduced in [9], for example arise in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure. BSDE provide the genuinely stochastic approach of control problems which find their analytical expression in the Hamilton–Jacobi–Bellman formalism. BSDE with drivers of this type keep being a source of intensive research.

As for Monte Carlo methods to simulate random processes, numerical schemes for BSDE provide a robust method for simulating and approximating solutions of control problems. Much has been done in recent years to create schemes for BSDE with Lipschitz continuous drivers (see [2], [7] or [5] and the references therein). So far BSDE with drivers of quadratic growth resisted attempts to allow such schemes, which was the main motivation for this paper.

If the driver is Lipschitz continuous, following [2], the strategy to prove convergence of a numerical discretization combines two ingredients: regularity of the trajectories of the control process, and a convenient *a priori* estimate for the solution. The regularity result we refer to can be found in [12] or [13]. It allows us to establish the convergence order for the approximation of the control process.

Our approach for the case of drivers with quadratic growth consists in adding Zhang’s path regularity result to the toolbox of qgBSDEs and, independently of the extension, to answer the question of explicit convergence rates for the truncation procedure in the setting of qgBSDEs.

In a first step, we extend the path regularity result for the control process to the setting of qgBSDE. The methods we apply to achieve this goal rely crucially on the power of the stochastic calculus of variations. If (Y, Z) is the solution pair of a BSDE, it is well known that the trace of the first Malliavin derivative allows a description of Z by the formula $D_t Y_t = Z_t$, which in turn allows estimates of Z in the sup norm, provided an extra continuity result is established. To describe path regularity of Z efficiently, one also needs estimates of the Malliavin derivative of Z in the sup norm, whence second order Malliavin derivatives of Y are needed and add to the complexity of the treatment. We are able to derive the path regularity result without assuming hypotheses that imply boundedness of the Z process.

In the second step of our approach, we truncate the quadratic growth part of the driver to fall back into the setting of Lipschitz continuous drivers. We are able to explicitly capture the convergence rate for the solutions of the truncated BSDE as a function of the truncation height. Combining the error estimate for the truncation with the ones for the discretization in any existent numerical scheme for BSDE with Lipschitz continuous drivers, we find a numerical approximation for quadratic growth BSDE. This result does not depend on Zhang’s path regularity result but depend partially on the results that lead to it.

This result is new to the best of our knowledge. The truncation procedure, however, does not look like the most efficient solution one hopes for. The main drawback of the approach resides in the running times of the numerical algorithm. Roughly, if K is the truncation dependent Lipschitz constant, the time step h of the partition for the usual numerical discretization has to satisfy $e^K h < 1$ modulo some multiplicative constant which results from the use of Gronwall’s inequality. So if the truncation height increases, h will have to become small very quickly, which computationally is a rather inconvenient fact. At this stage we have to leave the question open if a method exists with a convergence rate that depends on the Lipschitz constant only in a polynomial fashion instead of an exponential one. Of course it is conceivable that such a method is based on a discretization of the underlying qgBSDE without the intermediate step of truncating the driver. However, we wish to point out that such a procedure has its difficulties. From our

experience, the discretization may be well defined and studied as the partition's mesh size tends to zero. But to show convergence to the original solution and to provide a convergence rate appear as very difficult problems that to date remain unsolved.

The paper is organized as follows. In the introductory Section 2 we recall some of the well-known results concerning SDE and BSDE. In Section 3 we establish some estimates concerning a special class of BSDE, and in Section 4 we establish the second order Malliavin differentiability of solutions of Lipschitz BSDE and qgBSDE. These results are used in Section 5 to state and prove several regularity results for the trajectories of the solution processes. In Section 6 we discuss convergence rates of solutions of truncated BSDE to those related to BSDE with drivers of quadratic growth.

2. Preliminaries

2.1. Spaces and notation

Throughout fix $T > 0$. We work on a canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a d -dimensional Wiener process $W = (W^1, \dots, W^d)$ restricted to the time interval $[0, T]$, and we denote by $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$ its natural filtration enlarged in the usual way by the \mathbb{P} -zero sets. We shall need the following operators, and auxiliary spaces of functions and stochastic processes: let $p \geq 2, m, n, d \in \mathbb{N}$, \mathbb{Q} a probability measure on (Ω, \mathcal{F}) . We use the symbol $\mathbb{E}^{\mathbb{Q}}$ for the expectation with respect to \mathbb{Q} , and omit the superscript for the canonical measure \mathbb{P} . For vectors $x = (x^1, \dots, x^m)$ in Euclidean space \mathbb{R}^m we write $|x| = (\sum_{i=1}^m (x^i)^2)^{\frac{1}{2}}$. By $\mathbb{1}_A$ we denote the indicator function of a set A . We denote further

- $C_b^k(\mathbb{R}^m)$ the set of k -times differentiable real valued maps defined on \mathbb{R}^m with bounded partial derivatives up to order k , and $C_b^\infty(\mathbb{R}^m) = \cap_{k \geq 1} C_b^k(\mathbb{R}^m)$; We omit the subscript b to denote the same set but without the boundedness assumptions.
- $B_n^{m \times d}$ the set of all functions $h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times d}$ for which there is a constant C such that for all $t \in [0, T]$ we have $|h(t, x)| \leq C(1 + |x|)$ and $x \mapsto h(t, x)$ is differentiable with bounded Lipschitz derivative.
- $L^p(\mathbb{R}^m; \mathbb{Q})$ the space of \mathcal{F}_T -measurable random variables $X : \Omega \mapsto \mathbb{R}^m$, normed by $\|X\|_{L^p} = \mathbb{E}^{\mathbb{Q}}[|X|^p]^{\frac{1}{p}}$; L^∞ the space of bounded random variables.
- $S^p(\mathbb{R}^m)$ the space of all measurable processes $(Y_t)_{t \in [0, T]}$ with values in \mathbb{R}^m normed by $\|Y\|_{S^p} = \mathbb{E}[(\sup_{t \in [0, T]} |Y_t|)^p]^{\frac{1}{p}}$; $S^\infty(\mathbb{R}^m)$ the space of bounded measurable processes.
- $\mathcal{H}^p(\mathbb{R}^m, \mathbb{Q})$ the space of all progressively measurable processes $(Z_t)_{t \in [0, T]}$ with values in \mathbb{R}^m normed by $\|Z\|_{\mathcal{H}^p} = \mathbb{E}^{\mathbb{Q}}[(\int_0^T |Z_s|^2 ds)^{p/2}]^{\frac{1}{p}}$.
- $BMO(\mathbb{Q})$ or $BMO_2(\mathbb{Q})$ the space of square integrable martingales Φ with $\Phi_0 = 0$ and satisfying

$$\|\Phi\|_{BMO(\mathbb{Q})}^2 = \sup_{\tau} \left\| \mathbb{E}^{\mathbb{Q}}[\langle \Phi \rangle_T - \langle \Phi \rangle_\tau | \mathcal{F}_\tau] \right\|_\infty < \infty,$$

where the supremum is taken over all stopping times $\tau \in [0, T]$.

- $\mathbb{D}^{k,p}(\mathbb{R}^d)$ and $\mathbb{L}_{k,d}(\mathbb{R}^d)$ are the spaces of Malliavin differentiable random variables and processes; see Section 2.2.

If there is no ambiguity about the underlying spaces or measures, we also omit them as arguments in the function spaces defined above.

To denote stochastic integral processes of the Wiener process on $[0, T]$, according to Paul–André Meyer, we write

$$Z * W = \int_0^\cdot Z_s dW_s, \quad \text{with } Z \in \mathcal{H}^2.$$

Constants appearing in inequalities of our proofs will for simplicity be denoted by C , although they may change from line to line.

2.2. Malliavin calculus

We shall use techniques of the stochastic calculus of variations. To this end, we use the following notation. For more details, we refer the reader to [11]. Let \mathcal{S} be the space of random variables of the form

$$\xi = F \left(\left(\int_0^T h_s^{1,i} dW_s^1 \right)_{1 \leq i \leq n}, \dots, \left(\int_0^T h_s^{d,i} dW_s^d \right)_{1 \leq i \leq n} \right),$$

where $F \in C_b^\infty(\mathbb{R}^{n \times d})$, $h^1, \dots, h^n \in L^2([0, T]; \mathbb{R}^d)$, $n \in \mathbb{N}$. To simplify the notation, we assume that all h^j are written as row vectors. For $\xi \in \mathcal{S}$, we define $D = (D^1, \dots, D^d) : \mathcal{S} \rightarrow L^2(\Omega \times [0, T])^d$ by

$$D_\theta^i \xi = \sum_{j=1}^n \frac{\partial F}{\partial x_{i,j}} \left(\int_0^T h_t^1 dW_t, \dots, \int_0^T h_t^n dW_t \right) h_\theta^{i,j}, \quad 0 \leq \theta \leq T, \quad 1 \leq i \leq d,$$

and for $k \in \mathbb{N}$ and $\theta = (\theta_1, \dots, \theta_k) \in [0, T]^k$ its k -fold iteration by

$$D_\theta^{(k)} = (D_{\theta_1}^{i_1} \cdots D_{\theta_k}^{i_k})_{1 \leq i_1, \dots, i_k \leq d}.$$

For $k \in \mathbb{N}$, $p \geq 1$ let $\mathbb{D}^{k,p}$ be the closure of \mathcal{S} with respect to the norm

$$\|\xi\|_{k,p}^p = \left\{ \|\xi\|_{L^p}^p + \sum_{i=1}^k \|D^{(i)} \xi\|_{(\mathcal{H}^p)^i}^p \right\}.$$

$D^{(k)}$ is a closed linear operator on the space $\mathbb{D}^{k,p}$. Observe that if $\xi \in \mathbb{D}^{1,2}$ is \mathcal{F}_t -measurable then $D_\theta \xi = 0$ for $\theta \in (t, T]$. Further denote $\mathbb{D}^{k,\infty} = \cap_{p \geq 1} \mathbb{D}^{k,p}$.

We also need Malliavin's calculus for smooth stochastic processes with values in \mathbb{R}^m . For $k \in \mathbb{N}$, $p \geq 1$, denote by $\mathbb{L}_{k,p}(\mathbb{R}^m)$ the set of \mathbb{R}^m -valued progressively measurable processes $u = (u^1, \dots, u^m)$ on $[0, T] \times \Omega$ such that

- (i) For Lebesgue a.a. $t \in [0, T]$, $u(t, \cdot) \in (\mathbb{D}^{k,p})^m$;
- (ii) $[0, T] \times \Omega \ni (t, \omega) \mapsto D^{(k)} u(t, \omega) \in (L^2([0, T]^{1+k}))^{d \times n}$ admits a progressively measurable version;
- (iii) $\|u\|_{k,p}^p = \left\{ \|u\|_{\mathcal{H}^p}^p + \sum_{i=1}^k \|D^{(i)} u\|_{(\mathcal{H}^p)^{1+i}}^p \right\} < \infty$.

For instance, for a process $X \in \mathbb{L}_{2,2}(\mathbb{R})$ we have

$$\begin{aligned} \|X\|_{1,2}^2 &= \mathbb{E} \left[\int_0^T |X_t|^2 dt + \int_0^T \int_0^T |D_\theta X_t|^2 d\theta dt \right], \\ \|X\|_{2,2}^2 &= \|X\|_{1,2}^2 + \mathbb{E} \left[\int_0^T \int_0^T \int_0^T |D_{\theta_1} D_{\theta_2} X_t|^2 d\theta_1 d\theta_2 dt \right]. \end{aligned}$$

Note that Jensen's inequality gives for all $p \geq 2$

$$\mathbb{E} \left[\left(\int_0^T \int_0^T |D_u X_t|^2 du dt \right)^{\frac{p}{2}} \right] \leq T^{p/2-1} \int_0^T \|D_u X\|_{\mathcal{H}^p}^p du.$$

This inequality is very useful since the techniques used to deal with BSDE do not allow a direct estimate of the left-hand side, but easily give access to the right-hand side.

Occasionally we shall work with processes taking their values already in a Hilbert space, for instance if we talk about Malliavin derivatives. We therefore have to generalize the Sobolev spaces defined above somewhat. For a Hilbert space H we start with elementary H -valued variables of the form $\xi = \sum_{i=1}^n \xi_i h_i$, where for $1 \leq i \leq n$ the variable ξ_i is of the form discussed above, and $h_i \in H$. We define similarly $D_\theta^j \xi = \sum_{i=1}^n D_\theta^j \xi_i h_i$, $0 \leq \theta \leq T$, $1 \leq j \leq d$, and higher derivatives by iteration. For $k \in \mathbb{N}$, $p \geq 1$ we then let $\mathbb{D}^{k,p}(H)$ be the closure of this set of elementary processes with respect to the norm

$$\|\xi\|_{k,p,H}^p = \left\{ \|\xi\|_H^p + \sum_{i=1}^k \|D^{(i)} \xi\|_{H \otimes (\mathcal{H}^p)^i}^p \right\}.$$

$D^{(k)}$ is a closed linear operator on the space $\mathbb{D}^{k,p}(H)$. In a similar way we define the spaces $\mathbb{L}_{k,p}(\mathbb{R}^m \times H)$.

We state an extension of Lemma 1.2.3 from [11]. This extension will play a crucial role in our proof of Malliavin differentiability.

Lemma 2.1. *Let H be a Hilbert space, $(F^n)_{n \geq 1}$ a sequence of random variables with values in H that converges to an H -valued process F in $L^2(\Omega \times H)$ and such that*

$$\sup_{n \in \mathbb{N}} \|DF^n\|_{H \otimes \mathcal{H}^2(\Omega \times [0,T])} < \infty.$$

Then F belongs to $\mathbb{L}^{1,2}(\mathbb{R} \times H)$, and the sequence of derivatives $(DF^n)_{n \in \mathbb{N}}$ converges to DF in the weak topology of $H \otimes \mathcal{H}^2(\Omega \times [0,T])$.

Proof. The proof of this lemma is analogous to the proof of Lemma 1.2.3 of [11], being based on the closedness of the Malliavin derivative operator. \square

2.3. Some results on BMO martingales

BMO martingales play a key role for a priori estimates needed in our sensitivity analysis of solutions of BSDE. For details about their theory we refer the reader to [8].

If Φ is a square integrable martingale with $\Phi_0 = 0$, the martingale representation theorem yields a square integrable process ϕ such that $\Phi_t = \int_0^t \phi_s dW_s$, $t \in [0, T]$. Hence the $BMO(\mathbb{Q})$ norm can be alternatively expressed as

$$\sup_{\tau \in [0,T]} \mathbb{E}^{\mathbb{Q}} \left[\int_\tau^T \phi_s^2 ds \mid \mathcal{F}_\tau \right] < \infty.$$

As an easy consequence, if $\Phi \in BMO$ then $\int H d\Phi \in BMO$ for any bounded adapted process H .

Lemma 2.2 (Properties of BMO Martingales). *Let Φ be a BMO martingale. Then we have:*

(1) *The stochastic exponential $\mathcal{E}(\Phi)$ is uniformly integrable.*

- (2) There exists a number $r > 1$ such that $\mathcal{E}(\Phi_T) \in L^r$. This property follows from the Reverse Hölder inequality. The maximal r with this property can be expressed explicitly in terms of the BMO norm of Φ . There exists as well an upper bound for $\|\mathcal{E}(\Phi_T)\|_{L^r}^r$ depending only on T , r and the BMO norm of Φ .
- (3) For probability measures \mathbb{P} and \mathbb{Q} satisfying $d\mathbb{Q} = \mathcal{E}(\Phi_T)d\mathbb{P}$, and for $\Phi \in BMO(\mathbb{P})$, the process $\hat{\Phi} = \Phi - \langle \Phi \rangle$ is a $BMO(\mathbb{Q})$ martingale.
- (4) Energy inequalities imply the inclusion $BMO \subset \mathcal{H}^p$ for all $p \geq 1$. More precisely, for $\Phi = \int_0^\cdot \phi_s ds \in BMO$ with BMO norm C , and $p \geq 1$ the following estimate holds

$$\mathbb{E} \left[\left(\int_0^T |\phi_s|^2 ds \right)^p \right] \leq 2p!(4C^2)^p.$$

2.4. The setting and its assumptions

For functions b, σ, g and f , for $x \in \mathbb{R}^m$ and a d -dimensional Brownian motion W we intend to study the solution processes of the following system of forward–backward stochastic differential equations (with generators of quadratic growth (qgFBSDE)). For $t \in [0, T]$ they are given by

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad (1)$$

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, \Theta_s) ds, \quad (2)$$

with $\xi = g(X_T)$ and $\Theta_s = (X_s, Y_s, Z_s)$.

For the functions figuring in the above system of equations we hierarchically order the properties they will be assumed to satisfy.

HX0 There is a constant K such that $b, \sigma_i : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $1 \leq i \leq d$, are uniformly Lipschitz continuous with Lipschitz constant K , and $b(\cdot, 0)$ and $\sigma_i(\cdot, 0)$, $1 \leq i \leq d$, are bounded by K .

HX1 Hypothesis HX0 holds. For any $0 \leq t \leq T$ the functions $b(t, \cdot), \sigma_i(t, \cdot)$, $1 \leq i \leq d$, are differentiable and its derivatives are uniformly Lipschitz with Lipschitz constant K independent of t . In other words, $\sigma \in B_m^{m \times d}$ and $b \in B_m^{m \times 1}$. There exists a positive constant c such that

$$y^T \sigma(t, x) \sigma^T(t, x) y \geq c|y|^2, \quad x, y \in \mathbb{R}^m, \quad t \in [0, T]. \quad (3)$$

HX2 Hypothesis HX1 holds. There exists a positive constant K such that $b(t, \cdot) \in C_b^2(\mathbb{R}^m)$ and $\sigma(t, \cdot) \in C_b^2(\mathbb{R}^{m \times d})$ with second derivatives bounded by K .

HY0 There is a positive constant M such that $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is absolutely uniformly bounded by M , hence $|\xi| \leq M$. $f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an adapted measurable function, continuous in the space variables, for which there exists a positive constant M such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$

$$\begin{aligned} |f(t, x, y, z)| &\leq M(1 + |y| + |z|^2), \\ |f(t, x, y, z) - f(t, x, y', z')| &\leq M|y - y'| + M(1 + |z| + |z'|)|z - z'| \\ |f(t, x, y, z) - f(t, x', y, z)| &\leq M(1 + |y| + |z|^2)|x - x'|. \end{aligned}$$

HY1 Hypothesis HY0 holds. f is differentiable in (x, y, z) and there exists $M \in \mathbb{R}_+$ such that

$$|\nabla_x f(t, x, y, z)| \leq M(1 + |y| + |z|^2),$$

$$|\nabla_y f(t, x, y, z)| \leq M,$$

$$|\nabla_z f(t, x, y, z)| \leq M(1 + |z|).$$

$g : \mathbb{R}^m \rightarrow \mathbb{R}$ is a Lipschitz differentiable function satisfying $|\nabla g| \leq M$.

HY2 Hypothesis HY1 holds, $g \in C_b^2(\mathbb{R}^m)$. The driver f is twice differentiable with continuous second order derivatives. There exists an adapted process $(K_t)_{0 \leq t \leq T}$ belonging to $\mathcal{S}^{2p}(\mathbb{R})$ for all $p \geq 1$ such that for any $t \in [0, T]$ all second order derivatives of f at $(t, \theta_t) = (t, X_t, Y_t, Z_t)$ are a.s. dominated by K_t .

2.5. Some results on SDE

We recall the results on SDE known from the literature that are relevant for this work. We state our assumptions in the multidimensional setting. However, for ease of notation we present some formulas in the one dimensional case.¹

Theorem 2.3 (Moment Estimates for SDE). Assume that HX0 holds. Then (1) has a unique solution and the following moment estimates hold: for any $p \geq 2$ there exists a constant $C > 0$, depending only on T, K and p such that for any $x \in \mathbb{R}^m, s, t \in [0, T]$

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t|^p \right] \leq C \mathbb{E} \left[|x|^p + \int_0^T (|b(t, 0)|^p + |\sigma(t, 0)|^p) dt \right], \quad (4)$$

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |X_u - X_s|^p \right] \leq C \mathbb{E} \left[|x|^p + \sup_{0 \leq t \leq T} \{ |b(t, 0)|^p + |\sigma(t, 0)|^p \} \right] |t - s|^{p/2}. \quad (5)$$

Furthermore, given two different initial conditions $x, x' \in \mathbb{R}^m$ and denoting the respective solutions of (1) by X^x and $X^{x'}$, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^x - X_t^{x'}|^p \right] \leq C |x - x'|^p.$$

Theorem 2.4 (Classical Differentiability). Assume HX1 holds. Then the solution process X of (1) as a function of the initial condition $x \in \mathbb{R}^m$ is differentiable and satisfies for $t \in [0, T]$

$$\nabla X_t = I_m + \int_0^t \nabla b(s, X_s) \nabla X_s ds + \int_0^t \nabla \sigma(s, X_s) \nabla X_s dW_s, \quad (6)$$

where I_m denotes the $m \times m$ unit matrix. Moreover, ∇X_t as an $m \times m$ -matrix is invertible for any $t \in [0, T]$. Its inverse $(\nabla X_t)^{-1}$ satisfies an SDE and for any $p \geq 2$ there are positive constants C_p and c_p such that

$$\|\nabla X\|_{\mathcal{S}^p} + \|(\nabla X)^{-1}\|_{\mathcal{S}^p} \leq C_p \quad (7)$$

and

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |(\nabla X_u) - (\nabla X_s)|^p + \sup_{s \leq u \leq t} |(\nabla X_u)^{-1} - (\nabla X_s)^{-1}|^p \right] \leq c_p |t - s|^{p/2}. \quad (8)$$

¹ For a beautiful presentation of this subsection's theorems we refer the reader to [5].

Theorem 2.5 (Malliavin Differentiability). Under HX1, $X \in \mathbb{L}_{1,2}$ and its Malliavin derivative admits a version $(u, t) \mapsto D_u X_t$ satisfying for $0 \leq u \leq t \leq T$ the SDE

$$D_u X_t = \sigma(u, X_u) + \int_u^t \nabla b(s, X_s) D_u X_s ds + \int_u^t \nabla \sigma(s, X_s) D_u X_s dW_s.$$

Moreover, for any $p \geq 2$ there is a constant $C_p > 0$ such that for $x \in \mathbb{R}^m$ and $0 \leq v \leq u \leq t \leq s \leq T$

$$\begin{aligned} \|D_u X\|_{S^p}^p &\leq C_p(1 + |x|^p), \\ \mathbb{E}[|D_u X_t - D_u X_s|^p] &\leq C_p(1 + |x|^p)|t - s|^{\frac{p}{2}}, \\ \|D_u X - D_v X\|_{S^p}^p &\leq C_p(1 + |x|^p)|u - v|^{\frac{p}{2}}. \end{aligned}$$

By Theorem 2.4, we have the representation

$$D_u X_t = \nabla X_t (\nabla X_u)^{-1} \sigma(u, X_u) \mathbb{1}_{[0,u]}(t), \quad \text{for all } u, t \in [0, T].$$

If HX2 holds, then $DX \in \mathbb{L}_{1,2}$. For all $v, u, t \in [0, T]$, $D_v D_u X_t$ admits a version which solves for $0 \leq v \leq u \leq t \leq T$

$$\begin{aligned} D_v D_u X_t &= \nabla \sigma(u, X_u) D_v X_u + \nabla \sigma(v, X_v) D_u X_v \\ &\quad + \int_u^t \left[\nabla b(s, X_s) D_v D_u X_s + \Delta b(s, X_s) D_v X_s D_u X_s \right] ds \\ &\quad + \int_u^t \left[\nabla \sigma(s, X_s) D_v D_u X_s + \Delta \sigma(s, X_s) D_v X_s D_u X_s \right] dW_s. \end{aligned}$$

Furthermore, there exists a continuous version of $(D_v D_u X_t)_{v,u,t \in [0,T]}$ such that for all $0 \leq v, u \leq T$ and $p \geq 2$ we have

$$\|D_u D_v X\|_{S^p}^p \leq C_p(1 + |x|^{2p}).$$

For $0 \leq v, v' \leq u, u' \leq t \leq T$ we have

$$\|D_v D_u X - D_{v'} D_{u'} X\|_{S^p}^p \leq C_p \left(|v - v'|^\frac{p}{2} + |u - u'|^\frac{p}{2} \right).$$

2.6. Results on BSDE with drivers of quadratic growth

We next collect some results on qgBSDE. For their original versions or more information, we refer to [9,1,3,4].

Theorem 2.6 (Properties of qgBSDE). Under HY0, HX0, the system (1), (2) has a unique solution $(X, Y, Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{H}^2$. The norms of Y and Z depend only on T, K, M as given by assumption HY0.

The martingale $Z * W$ belongs to the space of BMO martingales, and hence $Z \in \mathcal{H}^p$ for all $p \geq 2$. The following estimate holds²:

$$\|Z * W\|_{BMO} \leq \frac{4 + 6M^2 T}{3M^2} \exp \left\{ 6M \|\xi\|_{L^\infty} + MT \right\} < \infty.$$

Remark 2.7. Following point (2) of Lemma 2.2, we define a pair (\bar{r}, \bar{q}) such that $1/\bar{r} + 1/\bar{q} = 1$ and $\mathcal{E}(Z * W) \in L^{\bar{r}}$.

² This inequality follows from applying Itô's formula to $e^{aY_t + bt}$ with an appropriate choice of a and b .

In the following, when discussing BMO martingales, an appearing exponent \bar{r} will always be used in this sense.

For more properties about BMO martingales in the setting of BSDE with drivers of quadratic growth we refer to Lemma 2.1 in [1].

The two differentiability results we now present can be found in [4]. These results are natural extensions of results proved in [1] or [3]. For further details, comments and complete proofs we refer to [4].

Theorem 2.8 (Classical Differentiability). *Suppose that HX1 and HY1 hold. Then for all $p \geq 2$ the solution processes (X^x, Y^x, Z^x) of the system (1), (2) with initial vector $x \in \mathbb{R}^m$ for the forward component belongs to $S^p \times S^p \times \mathcal{H}^p$. The application $\mathbb{R}^m \ni x \mapsto (X^x, Y^x, Z^x) \in S^p(\mathbb{R}^m) \times S^p(\mathbb{R}) \times \mathcal{H}^p(\mathbb{R}^d)$ is differentiable. The derivatives of X satisfy (6) while the derivatives of (Y, Z) satisfy the linear BSDE*

$$\nabla Y_t^x = \nabla g(X_T^x) \nabla X_T^x - \int_t^T \nabla Z_s^x dW_s + \int_t^T \langle \nabla f(s, \Theta_s^x), \nabla \Theta_s^x \rangle ds. \quad (9)$$

If HX2 and HY2 hold, then there exists a version of the solution $\Omega \times [0, T] \times \mathbb{R}^m \ni (\omega, t, x) \mapsto (X_t^x, Y_t^x, Z_t^x)(\omega) \in \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^d$, such that for almost all ω , X^x and Y^x are continuous in time and continuously differentiable in x .

Theorem 2.9 (Malliavin Differentiability). *Suppose that HX1 and HY1 hold. Then the solution processes (X, Y, Z) of system (1), (2) verify*

- for any $0 \leq t \leq T$, $x \in \mathbb{R}^m$ we have $(Y_t, Z_t) \in \mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$. X satisfies the statement of Theorem 2.5, and a version of $(D_u Y_t, D_u Z_t)_{0 \leq u, t \leq T}$ satisfies

$$\begin{aligned} D_u Y_t &= 0, \quad D_u Z_t = 0, \quad t < u \leq T, \\ D_u Y_t &= \nabla g(X_T) D_u X_T + \int_t^T \langle \nabla f(s, \Theta_s), D_u \Theta_s \rangle ds \\ &\quad - \int_t^T D_u Z_s dW_s, \quad t \in [u, T]. \end{aligned} \quad (10)$$

Moreover, $(D_t Y_t)_{0 \leq t \leq T}$ defined by the above equation is a version of $(Z_t)_{0 \leq t \leq T}$.

- the following representation holds for any $0 \leq u \leq t \leq T$ and $x \in \mathbb{R}^m$

$$\begin{aligned} D_u Y_t &= \nabla_x Y_t (\nabla_x X_u)^{-1} \sigma(u, X_u), \quad a.s., \\ Z_t &= \nabla_x Y_t (\nabla_x X_t)^{-1} \sigma(s, X_t), \quad a.s.. \end{aligned} \quad (11)$$

3. Inequalities for BSDE with stochastic Lipschitz conditions

In this section we look closely at BSDE with drivers that satisfy Lipschitz conditions with random Lipschitz constants. Our interest in this problem is motivated by the following observation. If we formally differentiate the driver of our original BSDE, we see that the essential term Z^2 produces a term of the form ZDZ . In this term we may consider the factor Z as a random growth rate of the factor DZ .

Let ζ be a random variable and f a measurable function. We consider the BSDE

$$U_t = \zeta - \int_t^T V_s dW_s + \int_t^T f(\cdot, s, U_s, V_s) ds, \quad t \in [0, T]. \quad (12)$$

We state a set of assumptions for ζ and f . For $p \geq 1$ we stipulate

(HA1) ζ is \mathcal{F}_T -adapted random variable and $\zeta \in L^{2p}(\mathbb{R})$.

(HA2) $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is product measurable and there exists a positive constant M and a positive predictable process H such that for all $u, u' \in \mathbb{R}$ and $v, v' \in \mathbb{R}^d$ we have

$$|f(\cdot, \cdot, u, v) - f(\cdot, \cdot, u', v')| \leq M|u - u'| + H \cdot |v - v'|,$$

and such that $H * W$ is a BMO martingale.

(HA3) $(f(\cdot, t, 0, 0))_{t \in [0, T]}$ is a measurable (\mathcal{F}_t) -adapted process satisfying such that for all $p \geq 1$ we have $\mathbb{E}[(\int_0^T |f(\cdot, s, 0, 0)|ds)^p] < \infty$.

Moreover, we assume that (U, V) is a solution of BSDE (12), and the constant \bar{r} is related to the BMO martingale $H * W$ as in Remark 2.7.

3.1. Moment estimates for BSDE with random Lipschitz constant

For the study of sensitivity properties of solutions of qgFBSDE, as seen in [1] or [3], it is convenient to consider BSDE with random Lipschitz constants. The moment estimates for this type of BSDE one finds in the two cited papers still leave space for improvements. A weakness of the results of [1], owed to the techniques used, is the lack of an estimate for $\|U\|_{\mathcal{S}^2}$. We next state an extended moment estimate, obtained by using ideas of [3].

Lemma 3.1. *Let (HA1) through (HA3) be satisfied and take $p \geq 1$. Let $\bar{r} > 1$ be such that $\mathcal{E}(H * W) \in L^{\bar{r}}(\mathbb{P})$. Then there exists a positive constant C , depending only on p, T, M and the BMO norm of $H * W$, such that with the conjugate exponent \bar{q} of \bar{r} we have*

$$\|U\|_{\mathcal{S}^{2p}}^{2p} + \|V\|_{\mathcal{H}^{2p}}^{2p} \leq C \mathbb{E} \left[|\zeta|^{2p\bar{q}^2} + \left(\int_0^T |f(\cdot, s, 0, 0)|ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}. \quad (13)$$

Proof. Assumption (HA2) states that the driver is Lipschitz continuous in u . We first use this hypothesis to simplify the BSDE. For $t \in [0, T]$ we define

$$a_t = \frac{f(\cdot, t, U_t, V_t) - f(\cdot, t, 0, V_t)}{U_t} 1_{\{U_t \neq 0\}} \quad \text{and} \quad e_t = \exp \left\{ \int_0^t a_s ds \right\}.$$

Under (HA2), namely the Lipschitz property of f in the first spatial variable, the process a is well defined and absolutely uniformly bounded by M . Hence e is bounded from above and from below by a positive constant. For $t \in [0, T]$ we further define

$$b_t = \frac{f(\cdot, t, 0, V_t) - f(\cdot, t, 0, 0)}{|V_t|^2} V_t 1_{\{V_t \neq 0\}}.$$

By (HA2), b is well defined and bounded in absolute value by the process H . Applying Itô's formula to $(e_t U_t)_{t \in [0, T]}$ we obtain

$$e_t U_t = e_T \zeta + \int_t^T e_s [f(\cdot, s, 0, 0) + b_s V_s] ds - \int_t^T e_s V_s dW_s.$$

We simplify the BSDE further by defining a new measure \mathbb{Q}^b for which $W^b = W - \int_0^\cdot b_s ds$ is a \mathbb{Q}^b -Brownian motion. The Radon–Nikodym density of \mathbb{Q}^b with respect to \mathbb{P} is given by the

stochastic exponential $\mathcal{E}(b * W)$. Since $|b| \leq H$ we have $\|b * W\|_{BMO} \leq \|H * W\|_{BMO}$. Hence the measure \mathbb{Q}^b is indeed a probability measure. For $t \in [0, T]$ our BSDE takes the form

$$e_t U_t = e_T \zeta + \int_t^T e_s f(\cdot, s, 0, 0) ds - \int_t^T e_s V_s dW_s^b. \quad (14)$$

We now proceed with moment estimates. Taking conditional expectations with respect to \mathbb{Q}^b , estimating by absolute values and integrating on the whole interval we obtain

$$|e_t U_t| \leq \mathbb{E}^{\mathbb{Q}^b} \left[e_T |\zeta| + \int_0^T e_s |f(\cdot, s, 0, 0)| ds \middle| \mathcal{F}_t \right].$$

Applying Doob's moment inequality for $2p \geq 2$ we obtain a similar inequality as in this Theorem's statement, but under the measure \mathbb{Q}^b , i.e.

$$\|U\|_{S^{2p}(\mathbb{Q}^b)}^{2p} \leq C \mathbb{E}^{\mathbb{Q}^b} \left[|\zeta|^{2p} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p} \right]. \quad (15)$$

If we rewrite Eq. (14), isolate the stochastic integral on the left-hand side and take $t = 0$, use Burkholder–Davis–Gundy's inequality and remember that e is also bounded from below by a positive constant thanks to (HA2), we get

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^b} \left[\left(\int_0^T |V_s|^2 ds \right)^p \right] &\leq c_p \mathbb{E}^{\mathbb{Q}^b} \left[|e_T \zeta|^{2p} + \sup_{0 \leq t \leq T} |e_t U_t|^{2p} + \left(\int_0^T |e_s f(\cdot, s, 0, 0)| ds \right)^{2p} \right] \\ &\leq C \mathbb{E}^{\mathbb{Q}^b} \left[|\zeta|^{2p} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p} \right]. \end{aligned}$$

For the second inequality we used (15) and the fact that $\left[\int_0^T |V_s|^2 ds \right]^{1/2}$ is integrable.

Summing the last two inequalities we get

$$\|U\|_{S^{2p}(\mathbb{Q}^b)}^{2p} + \|V\|_{\mathcal{H}^{2p}(\mathbb{Q}^b)}^{2p} \leq C \mathbb{E}^{\mathbb{Q}^b} \left[|\zeta|^{2p} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p} \right]. \quad (16)$$

This inequality is already close to the one we have to deduce. To complete the proof, we just have to get rid of the dependence on \mathbb{Q}^b in the terms of the inequality. We do this for (15), noting that for the other inequality the arguments are very similar. As mentioned before, b is dominated by H and therefore

$$\|b * W\|_{BMO} \leq \|H * W\|_{BMO}.$$

Further, part (3) of Lemma 2.2 implies that since $b * W \in BMO(\mathbb{P})$, also $(-b) * W^b \in BMO(\mathbb{Q}^b)$. Moreover, since $[\mathcal{E}(b * W)]^{-1} = \mathcal{E}((-b) * W^b)$, part (2) of the same lemma states the existence³ of a real number $\bar{r} > 1$ for which $\mathcal{E}(b * W) \in L^{\bar{r}}(\mathbb{P})$ and $[\mathcal{E}(b * W)]^{-1} \in L^{\bar{r}}(\mathbb{Q}^b)$. The constant \bar{r} is estimated from the $BMO(\mathbb{P})$ norm of $H * W$, as indicated in Lemma 2.2.

Throughout let $D = \max\{\|\mathcal{E}(b * W)\|_{L^{\bar{r}}(\mathbb{P})}, \|\mathcal{E}(b * W)^{-1}\|_{L^{\bar{r}}(\mathbb{Q}^b)}\}$ and let \bar{q} be the conjugate Hölder exponent of \bar{r} .

³ Here we follow the notation we stipulated in Remark 2.7.

Combining (15) and Hölder's inequality, we obtain for any $p \geq 1$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \in [0, T]} |U_s|^{2p} \right] &= \mathbb{E}^{\mathbb{Q}^b} \left[\mathcal{E}(b * W)^{-1} \sup_{s \in [0, T]} |U_s|^{2p} \right] \leq D \mathbb{E}^{\mathbb{Q}^b} \left[\sup_{s \in [0, T]} |U_s|^{2p\bar{q}} \right]^{\frac{1}{\bar{q}}} \\ &\leq C_1 D \mathbb{E}^{\mathbb{Q}^b} \left[|\zeta|^{2p\bar{q}} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p\bar{q}} \right]^{\frac{1}{\bar{q}}} \\ &= C_1 D \mathbb{E}^{\mathbb{P}} \left[\mathcal{E}(b * W) \left(|\zeta|^{2p\bar{q}} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p\bar{q}} \right) \right]^{\frac{1}{\bar{q}}} \\ &\leq C_2 D^{\frac{1+\bar{q}}{\bar{q}}} \mathbb{E}^{\mathbb{P}} \left[|\zeta|^{2p\bar{q}^2} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}, \end{aligned}$$

where C_1, C_2 represent constants depending on p, M, T . Similarly, with another constant C_3 ,

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |V_s|^2 ds \right)^p \right] \leq C_3 D^{\frac{1+\bar{q}}{\bar{q}}} \mathbb{E}^{\mathbb{P}} \left[|\zeta|^{2p\bar{q}^2} + \left(\int_0^T |f(\cdot, s, 0, 0)| ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}.$$

Combining the two estimates we obtain (13). \square

3.2. A priori estimates for BSDE with random Lipschitz constant

In this section, following the results of the previous one, we derive a priori inequalities which serve in the usual way to compare solutions of BSDE of the type considered obtained for different system parameters such as initial states of the forward part. This result will later be used to determine the good candidates for the derivatives of our original qgBSDE.

For each $i \in \{1, 2\}$, let ζ_i be a random variable satisfying condition (HA1) and f_i a driver function satisfying (HA2) and (HA3) with respective square integrable processes H^i such that $H^i * W \in BMO$. With this random variable and driver function we investigate the following BSDE

$$U_t^{(i)} = \zeta_i - \int_t^T V_s^{(i)} dW_s + \int_t^T f_i(\omega, s, U_s^{(i)}, V_s^{(i)}) ds, \quad t \in [0, T]. \quad (17)$$

Lemma 3.2. Assume the conditions of Lemma 3.1 hold for (17). Take further \bar{q} with respect to the BSDE with $i = 1$. Then we have for any $p \geq 1$ a positive constant C exists such that

$$\begin{aligned} &\|U^{(1)} - U^{(2)}\|_{\mathcal{S}^{2p}}^{2p} + \|V^{(1)} - V^{(2)}\|_{\mathcal{H}^{2p}}^{2p} \\ &\leq C \mathbb{E} \left[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} + \left(\int_0^T |(f_1 - f_2)(\cdot, s, U_s^{(2)}, V_s^{(2)})| ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}, \end{aligned}$$

with \bar{q} given as in Remark 2.7 with respect to the BMO martingale $(H^1 * W)$.

Proof. The arguments to prove this inequality are similar to those used in the proof of Lemma 3.1. Therefore we will omit some of the already familiar details.

Define $\delta U = U^{(1)} - U^{(2)}$, $\delta V = V^{(1)} - V^{(2)}$, $\delta \zeta = \zeta_1 - \zeta_2$ and $\delta f(\cdot, t, u, v) = (f_1 - f_2)(\cdot, t, u, v)$. Then to simplify the BSDE define a and b for $t \in [0, T]$ by

$$a_t = \frac{f_1(\cdot, s, U_t^{(1)}, V_t^{(1)}) - f_1(\cdot, s, U_t^{(2)}, V_t^{(1)})}{U_t^{(1)} - U_t^{(2)}} \mathbb{1}_{\{U_t^{(1)} \neq U_t^{(2)}\}},$$

$$b_t = \frac{f_1(\cdot, s, U_t^{(2)}, V_t^{(1)}) - f_1(\cdot, s, U_t^{(2)}, V_t^{(2)})}{|V_t^{(1)} - V_t^{(2)}|^2} (V_t^{(1)} - V_t^{(2)}) \mathbb{1}_{\{V_t^{(1)} \neq V_t^{(2)}\}}.$$

We arrive at an equation similar to (14) given by:

$$e_t \delta U_t = e_T \delta \zeta + \int_t^T [e_s \delta f(\cdot, s, U_s^{(2)}, V_s^{(2)})] ds - \int_t^T \delta V_s dW_s^b,$$

with $W^b = W - \int_0^\cdot b_s ds$. Define \mathbb{Q}^b with respect to b as before. Now we may proceed as in the proof of Lemma 3.1. The existence of the integral of $\delta f(\cdot, s, U_s^{(2)}, V_s^{(2)})$ is justified by observing that we can dominate δf using our assumptions and also because Lemma 3.1 is applicable to each individual BSDE. The result follows. \square

Without prior knowledge of the form of f_1 and f_2 the right-hand side of the lemma's inequality cannot be treated further. In the following result we assume that the drivers satisfy a stochastic linearity property. Then the increment in the drivers can be further estimated.

Corollary 3.3. Assume the conditions of Lemma 3.2 are satisfied, and furthermore that for each $i \in \{1, 2\}$ the driver f_i is linear, i.e. it satisfies

$$f_i(\cdot, t, u, v) = \alpha_i(\cdot, t) + \beta_i(\cdot, t)u + \langle \gamma_i(\cdot, t), v \rangle,$$

with $(\alpha_i, \beta_i, \gamma_i)$ adapted random processes belonging to $\mathcal{H}^{2p}(\mathbb{R}) \times \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ for any $p \geq 1$. Moreover, we assume that β_i is bounded and that $(\gamma_i * W) \in BMO$. Then

$$\begin{aligned} \|U^{(1)} - U^{(2)}\|_{\mathcal{S}^{2p}}^{2p} + \|V^{(1)} - V^{(2)}\|_{\mathcal{H}^{2p}}^{2p} &\leq C \left\{ \mathbb{E} \left[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} \right. \right. \\ &\quad + \left(\int_0^T |\alpha_1(\cdot, s) - \alpha_2(\cdot, s)| ds \right)^{2p\bar{q}^2} \Big]^{\frac{1}{\bar{q}^2}} + \mathbb{E} \left[\left(\int_0^T |\beta_1(\cdot, s) - \beta_2(\cdot, s)| ds \right)^{4p\bar{q}^2} \right. \\ &\quad \left. \left. + \left(\int_0^T |\gamma_1(\cdot, s) - \gamma_2(\cdot, s)|^2 ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{2\bar{q}^2}} \right\}. \end{aligned}$$

Proof. Starting with the inequality of Lemma 3.2, and injecting the new assumptions, we obtain

$$\begin{aligned} \|U^{(1)} - U^{(2)}\|_{\mathcal{S}^{2p}}^{2p} + \|V^{(1)} - V^{(2)}\|_{\mathcal{H}^{2p}}^{2p} &\leq C \mathbb{E} \left[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} \right. \\ &\quad + \left(\int_0^T |\alpha_1(\cdot, s) - \alpha_2(\cdot, s)| ds \right)^{2p\bar{q}^2} + \left(\sup_{0 \leq t \leq T} |U_t^{(2)}| \right)^{2p\bar{q}^2} \\ &\quad \times \left(\int_0^T |\beta_1(\cdot, s) - \beta_2(\cdot, s)| ds \right)^{2p\bar{q}^2} \\ &\quad \left. + \left(\int_0^T |V_s^{(2)}|^2 ds \right)^{p\bar{q}^2} \left(\int_0^T |\gamma_1(\cdot, s) - \gamma_2(\cdot, s)|^2 ds \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}. \end{aligned}$$

The moment estimates of Lemma 3.1 ensure that $\|U^{(2)}\|_{\mathcal{S}^{4p\bar{q}^2}}$ and $\|V^{(2)}\|_{\mathcal{H}^{4p\bar{q}^2}}$ are finite. Hence a simple application of Hölder's inequality yields the desired result. \square

4. Second order Malliavin differentiability

We now give sufficient conditions on our system of stochastic equations which ensure that the solution processes are twice Malliavin differentiable.

4.1. The main result

Theorem 4.1. Assume HX2 and HY2 hold. Then the solution process $\Theta = (X, Y, Z)$ of the qgFBSDE (1), (2) is twice Malliavin differentiable, i.e. for each $u \in [0, T]$ and $i \in \{1, \dots, d\}$ the processes $(D_u^i Y_t, D_u^i Z_t) \in \mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$. A version of $\{(D_v^j D_u^i Y_t, D_v^j D_u^i Z_t); 0 \leq v \leq u \leq t \leq T\}$ with $0 \leq j, i \leq d$ satisfies

$$\begin{aligned} D_v^j D_u^i Y_t &= D_v^j D_u^i \xi - \int_t^T D_v^j D_u^i Z_s dW_s \\ &\quad + \int_t^T \left[(D_v^j \Theta_s)^T [Hf](s, \Theta_s) D_u^i \Theta_s + \langle \nabla f(s, \Theta_s), D_v^j D_u^i \Theta_s \rangle \right] ds, \end{aligned} \quad (18)$$

where $[Hf]$ is the Hessian matrix of the function f and $\xi = g(X_T)$. Considered as a BSDE, (18) admits a unique solution.

Moreover $\{D_t D_u Y_t; 0 \leq u \leq t \leq T\}$ is a version of $\{D_u Z_t; 0 \leq u \leq t \leq T\}$.

By Theorem 2.5, condition HX2 already implies that $X \in \mathbb{L}_{2,p}$. Therefore one only needs to prove the Malliavin differentiability of (DY, DZ) .

4.2. Strategy of the proof

The main problem in proving the variational differentiability of Eq. (10) is given by the growth of $\nabla_z f(\cdot, z)$ in z . HY1 states that $\nabla_z f(\cdot, z)$ is dominated by $C(1 + |z|)$. Considering (10) as a BSDE with solution process (DY, DZ) leads to interpreting the influence of $\nabla_z f(\cdot, z)$ in the driver as a random Lipschitz constant. We aim at using the same strategy of proof as in [1]: we approximate the BSDE (10) by truncating the random Lipschitz constant, and then use Lemma 2.1 to obtain variational differentiability in the limit. Therefore we mainly have to establish the conditions of Lemma 2.1.

4.2.1. A differentiable truncation family for the identity function

We start by introducing a sequence of smooth real valued functions $(\tilde{h}_n)_{n \in \mathbb{N}}$ that truncate the identity on the real line and that will be used to truncate the variable z in the function $\nabla_z f(\cdot, \cdot, \cdot, z)$. We choose $\tilde{h}_n : \mathbb{R} \rightarrow \mathbb{R}$ continuously differentiable with the following properties:

- $(\tilde{h}_n)_{n \in \mathbb{N}}$ converges locally uniformly to the identity; For all $n \in \mathbb{N}$ and $z \in \mathbb{R}$ it holds that $|\tilde{h}_n(z)| \leq |z|$, $|\tilde{h}_n'(z)| \leq n + 1$ and

$$\tilde{h}_n(z) = \begin{cases} (n+1), & z > n+2, \\ z, & |z| \leq n, \\ -(n+1), & z < -(n+2). \end{cases} \quad (19)$$

- The derivative of \tilde{h}_n is absolutely bounded by 1, and converges to 1 locally uniformly.

We remark that such sequence of functions exists. The above requirements are for instance consistent with

$$\tilde{h}_n(z) = \begin{cases} (-n^2 + 2nz - z(z-4))/4, & z \in [n, n+2], \\ (n^2 + 2nz + z(z+4))/4, & z \in [-(n+2), -n]. \end{cases}$$

We then define $h_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $z \mapsto h_n(z) = (\tilde{h}_n(z_1), \dots, \tilde{h}_n(z_d))$, $n \in \mathbb{N}$.

4.2.2. The family of truncated FBSDE and results concerning them

Recall the notation $\Theta = (X, Y, Z)$ for the solution of system (1), (2), the driver of BSDE (10) with terminal condition $\xi = g(X_T)$, where g is a bounded differentiable function and **HX1** is satisfied. For $n \in \mathbb{N}$ take the sequence $(h_n)_{n \in \mathbb{N}}$ defined in (19) and define the sequence of approximate drivers $F^n : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$F^n(t, x, u, v) = \langle \nabla_x f(t, \Theta_t), x \rangle + \nabla_y f(t, \Theta_t)u + \langle \nabla_z f(t, X_t, Y_t, h_n(Z_t)), v \rangle. \quad (20)$$

The advantage of approximating the driver in this way is a technical one: we can make use of the well-known Θ and its properties, and do not have to deal with approximations of Θ and its Malliavin derivatives at the same time.

For $i \in \{1, \dots, d\}$, $0 \leq u \leq t \leq T$ and $n \in \mathbb{N}$ consider the following BSDE

$$U_{u,t}^n = D_u^i \xi + \int_t^T F^n(s, \Xi_{u,s}^n) ds - \int_t^T V_{u,s}^n dW_s, \quad \Xi_{u,s}^n = (D_u^i X_s, U_{u,s}^n, V_{u,s}^n), \quad (21)$$

where $D^i \xi$, $D^i X$ denote the first Malliavin derivatives of ξ and X respectively.

In the following lemma we state existence, uniqueness and Malliavin differentiability of the solution processes of BSDE (21). The lemma's proof will result from a theorem formulated in the [Appendix](#), where all the hypotheses, variants of the hypotheses employed in [Theorem 4.1](#), are formulated. To avoid repetitions, we do not formulate them here again.

Lemma 4.2 (Second Order Malliavin Diff. of Lipschitz BSDE). *For each $n \in \mathbb{N}$, (21) has a unique solution (U^n, V^n) in $S^{2p}([0, T] \times [0, T]) \times \mathcal{H}^{2p}([0, T] \times [0, T])$ for any $p \geq 1$.*

Furthermore for $0 \leq u \leq t \leq T$ the random variables $(U_{u,t}^n, V_{u,t}^n)$ are Malliavin differentiable and for any $j \in \{1, \dots, d\}$ a version of $\{(D_v^j U_{u,t}^n, D_v^j V_{u,t}^n); 0 \leq v \leq u \leq t \leq T\}$ satisfies

$$\begin{aligned} D_v^j U_{u,t}^n &= D_v^j D_u^i \xi - \int_t^T D_v^j V_{u,s}^n dW_s \\ &\quad + \int_t^T \left[(D_v^j F^n)(s, \Xi_{u,s}^n) + \langle (\nabla F^n)(s, \Xi_{u,s}^n), D_v^j \Xi_{u,s}^n \rangle \right] ds, \end{aligned} \quad (22)$$

with $\Xi_{u,s}^n = (D_u^i X_s, U_{u,s}^n, V_{u,s}^n)$ and $D_v^j \Xi_{u,s}^n = (D_v^j D_u^i X_s, D_v^j U_{u,s}^n, D_v^j V_{u,s}^n)$, $0 \leq v \leq u \leq s \leq T$.

For clarity of exposition we give a few words about the driver of BSDE (22). Assuming $d = m = 1$ and hence omitting the superscripts i and j , and denoting $\Theta = (X, Y, Z)$ and $\Theta^n = (X, Y, h_n(Z))$, we can describe the first term inside the integral by

$$\begin{aligned} (D_v F^n)(t, \Xi_{u,t}^n) &= D_v[(\nabla_x f)(t, \Theta_t)] D_u X_t + D_v[(\nabla_y f)(t, \Theta_t)] U_{u,t}^n \\ &\quad + D_v[(\nabla_z f)(t, \Theta_t^n)] V_{u,t}^n. \end{aligned}$$

These three terms can be further specified by

$$\begin{aligned} D_v[(\nabla_x f)(t, \Theta_t)] D_u X_t &= (\nabla_{xx} f)(t, \Theta_t) D_v X_t D_u X_t + (\nabla_{xy} f)(t, \Theta_t) D_v Y_t D_u X_t \\ &\quad + (\nabla_{xz} f)(t, \Theta_t) D_v Z_t D_u X_t, \end{aligned}$$

an analogous expression for $D_v[(\nabla_y f)]$, while the last part is given by

$$D_v[(\nabla_z f)(t, \Theta_t^n)]V_{u,t}^n = (\nabla_{zx} f)(t, \Theta_t^n)D_v X_t V_{u,t}^n + (\nabla_{zy} f)(t, \Theta_t^n)D_v Y_t V_{u,t}^n \\ + (\nabla_{zz} f)(t, \Theta_t^n)h'_n(Z_t)D_v Z_t V_{u,t}^n.$$

The second term of the driver in (22) can be expressed by

$$\langle (\nabla F^n)(s, \Xi_{u,s}^n), D_v \Xi_{u,s}^n \rangle = \nabla_x f(s, \Theta_s)D_v D_u X_s + \nabla_y f(s, \Theta_s)D_v U_{u,s}^n \\ + \nabla_z f(s, X_s, Y_s, h_n(Z_s))D_v V_{u,s}^n.$$

To compact notation a bit, we denote the driver component in (22) not containing $D_v U_{u,s}^n$ and $D_v V_{u,s}^n$ by

$$A_{v,u,s}^n = (D_v F^n)(s, \Xi_{u,s}^n) + (\nabla_x F^n)(s, \Xi_{u,s}^n)D_v D_u X_s, \quad v, u, s \in [0, T]. \quad (23)$$

Before giving the proofs of Theorem 4.1 or Lemma 4.2 we prove two helpful lemmas.

Remark 4.3. Since $|h_n(z)|$ is dominated by $|z|$, it is clear from HY1 that

$$\sup_{n \in \mathbb{N}} \|\nabla_z f(t, X, Y, h_n(Z)) * W\|_{BMO} \leq C \|(1 + |Z|) * W\|_{BMO} < \infty.$$

Hence by Lemma 2.2, there exists an \bar{r} such that the stochastic exponentials related to the two BMO martingales above belong both to $L^{\bar{r}}$. We remark that \bar{r} is independent of n .

Lemma 4.4. Assume HX2 and HY2 hold, that (U^n, V^n) solve BSDE (21) and (DY, DZ) BSDE (10). Then, for any $p \geq 1$ we have

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \left\{ \mathbb{E} \left[\left(\int_0^T |D_u Y_s|^2 + |D_u Z_s|^2 ds \right)^p + \left(\int_0^T |U_{u,s}^n|^2 + |V_{u,s}^n|^2 ds \right)^p \right] \right\} < \infty.$$

Proof. For any $n \in \mathbb{N}, z \in \mathbb{R}$ our hypothesis gives $|h_n(z)| \leq |z|$. Hence the driver F^n of (20) satisfies the same growth conditions as the driver of BSDE (10). Therefore, one can apply the results of Section 3 to either BSDE and obtain for $p \geq 1$

$$\sup_{0 \leq u \leq T} \left\{ \mathbb{E} \left[\left(\int_0^T |D_u Y_s|^2 ds \right)^p + \left(\int_0^T |D_u Z_s|^2 ds \right)^p + \left(\int_0^T |U_{u,s}^n|^2 ds \right)^p \right. \right. \\ \left. \left. + \left(\int_0^T |V_{u,s}^n|^2 ds \right)^p \right] \right\} \leq C \sup_{0 \leq u \leq T} \mathbb{E} \left[\left(|D_u \xi|^2 + \int_0^T |\nabla_x f(s, \Theta_s)D_u X_s|^2 ds \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}},$$

with \bar{q} the Hölder conjugate of \bar{r} . The results of Section 2.5 combined with assumptions HX2 and HY2 yield the finiteness of the right-hand side of the inequality. \square

Lemma 4.5. Assume HX2 and HY2 hold. For all $p \geq 1$ we have

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq u, v \leq T} \mathbb{E} \left[|D_v D_u \xi|^{2p} + \left(\int_0^T |A_{v,u,s}^n| ds \right)^{2p} \right] < \infty,$$

with $A^n, n \in \mathbb{N}$, given by (23).

Proof. To prove this result we analyze each term in more detail.

Part (1): The first term presents little difficulty, since $\xi = g(X_T)$ and X is a diffusion process. For $0 \leq v \leq u \leq T$ we have $D_v D_u \xi = D_v X_T [Hg](X_T) D_u X_T + \nabla g(X_T) D_v D_u X_T$, where $[Hg]$ is the Hessian matrix of g .

Since $g \in C_b^2$, we may use the inequality $2ab \leq (a^2 + b^2)$ valid for $a, b \in \mathbb{R}$ combined with Theorem 2.5 to obtain

$$\sup_{0 \leq u, v \leq T} \mathbb{E} \left[|D_v D_u \xi|^{2p} \right] \leq C \sup_{0 \leq u, v \leq T} \mathbb{E} \left[|D_u X_T|^{4p} + |D_v D_u X_T|^{2p} \right] < \infty.$$

Part (2): We now analyze the second term, starting with the identification

$$\begin{aligned} A_{v,u,s}^n &= (D_v F^n)(s, \Xi_{u,s}^n) + (\nabla_x F^n)(s, \Xi_{u,s}^n) D_v D_u X_s \\ &= (D_v F^n)(s, \Xi_{u,s}^n) + (\nabla_x f)(s, \Theta_s) D_v D_u X_s, \end{aligned}$$

for $v, u, s \in [0, T]$. Now $(D_v F^n)(t, \Xi_{u,s}^n)$ is composed of products of first order Malliavin derivatives of X, Y or Z and second order partial derivatives of f . Assumption HY2 guarantees that the second order derivatives of f are dominated by a process K belonging to $\mathcal{S}^{2p}([0, T])$. Combining this with the hypothesis $|h'_n| \leq 1$ for all n we easily obtain

$$\begin{aligned} |(D_v F^n)(s, \Xi_{u,s}^n)| &\leq C K_s \left\{ |D_v X_s|^2 + |D_u X_s|^2 + |D_v Y_s|^2 + |U_{u,s}^n|^2 \right. \\ &\quad \left. + |D_v Z_s|^2 + |V_{u,s}^n|^2 \right\}. \end{aligned}$$

Summands involving the Malliavin derivatives of X can be dealt with arguments as in *part (1)* of this proof. Furthermore,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} |K_t| \int_0^T \left[|U_{u,s}^n|^2 + |V_{u,s}^n|^2 + |D_u Y_s|^2 + |D_u Z_s|^2 \right] ds \right)^{2p} \right] \\ \leq \|K\|_{\mathcal{S}^{4p}}^{2p} \sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \| |U_u^n| + |V_u^n| + |D_u Y| + |D_u Z| \|_{\mathcal{H}^{8p}}^{4p} < \infty. \end{aligned}$$

The last inequality is satisfied by Lemma 4.4 and the fact that $K \in \mathcal{S}^{2p}$ for all $p \geq 1$.

We are left with the analysis of the term $(\nabla_x f)(s, \Theta_s) D_v D_u X_s$. From condition HY1, $\nabla_x f(s, \Theta_s)$ is dominated by $M(1 + |Y_s| + |Z_s|^2)$ with a bounded process Y , and so we obtain

$$\begin{aligned} \sup_{0 \leq u, v \leq T} \mathbb{E} \left[\left(\int_0^T |(\nabla_x f)(s, \Theta_s) D_v D_u X_s| ds \right)^{2p} \right] \\ \leq C \sup_{0 \leq u, v \leq T} \mathbb{E} \left[\sup_{t \in [0, T]} |D_v D_u X_t|^{2p} \left(\int_0^T (1 + |Z_s|^2) ds \right)^{2p} \right] \\ \leq C \sup_{0 \leq u, v \leq T} \|D_v D_u X\|_{\mathcal{S}^{4p}}^{2p} \|1 + |Z|\|_{\mathcal{H}^{8p}}^{4p} < \infty. \end{aligned}$$

For the last two inequalities we used Hölder's inequality, that $Z \in \mathcal{H}^{2p}$ for all $p \geq 1$ and Theorem 2.5.

The lemma's inequality follows from a combination of *Part (1)* and *Part (2)*. \square

We are now in a position to prove Lemma 4.2. We will use Theorem A.1 stated in the Appendix.

Proof of Lemma 4.2. We have to establish the hypotheses to hold for the application of Theorem A.1. The terminal condition is given by the Malliavin derivative of $\xi = g(X_T)$ with $g \in C_b^2$. In view of Theorems 2.4 and 2.5, conditions (A2) and (A4) are satisfied.

Given our construction, it is clear that for each fixed $n \in \mathbb{N}$, the driver F^n is uniformly Lipschitz continuous in (y, z) , since $\nabla_y f$ and $\nabla_z f(\cdot, h_n(\cdot))$ are bounded. The boundedness of

$\nabla_x f$ combined with the fact that $DX \in \mathcal{S}^{2p}([0, T] \times [0, T])$ enables us to conclude

$$\sup_{0 \leq u \leq T} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\nabla_x f(t, \Theta_t) D_u X_t|^{2p} \right] < \infty,$$

and hence condition (A1) is also satisfied.

The verification of condition (A3) is also simple. F^n is continuous differentiable in (y, z) . Furthermore since Y and Z are Malliavin differentiable and X is twice Malliavin differentiable, we have that $F^n(t, D_u X_t, 0, 0)$, $F^n(t, 0, 1, 0)$ and $F^n(t, 0, 0, 1)$ are also Malliavin differentiable for $0 \leq u \leq t \leq T$. The proof of the moment inequality of assumption (A3) is a consequence of Lemma 4.5.

Hence we may apply Theorem A.1. \square

4.3. Proof of Theorem 4.1

We are finally able to prove the main result of this section.

Proof of Theorem 4.1. To prove this result we apply Lemma 2.1. We have to show that the Lemma's assumptions are satisfied. Fix $0 \leq u \leq t \leq T$.

(1) Lemma 4.2 ensures existence, uniqueness and Malliavin differentiability of each $(U_{u,t}^n, V_{u,t}^n)$.

(2) We now prove the \mathcal{H}^2 -convergence of $(U_{u,\cdot}^n, V_{u,\cdot}^n)$ to $(D_u Y, D_u Z)$. Using Lemma 3.2 applied to the BSDE resulting from the difference $D_u Y - U_{u,\cdot}^n$ (see BSDE (10) and (21)), we have with $\Theta = (X, Y, Z)$ and $\Theta^n = (X, Y, h_n(Z))$

$$\begin{aligned} & \sup_{0 \leq u \leq T} \mathbb{E} \left[\int_0^T |D_u Y_s - U_{u,s}^n|^2 ds + \int_0^T |D_u Z_s - V_{u,s}^n|^2 ds \right] \\ & \leq C \sup_{0 \leq u \leq T} \mathbb{E} \left[\left(\int_0^T |\nabla_z f(s, \Theta_s) - \nabla_z f(s, \Theta_s^n)| |V_{u,s}^n| ds \right)^{2\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}} \\ & \leq C \sup_{0 \leq u \leq T} \mathbb{E} \left[\left(\int_0^T |V_{u,s}^n|^2 ds \right)^{2\bar{q}^2} \right]^{\frac{1}{2\bar{q}^2}} \mathbb{E} \left[\left(\int_0^T |\nabla_z f(s, \Theta_s) - \nabla_z f(s, \Theta_s^n)|^2 ds \right)^{2\bar{q}^2} \right]^{\frac{1}{2\bar{q}^2}}, \end{aligned}$$

where \bar{q} is related to the BMO martingale $(\nabla_z f(\cdot, X, Y, Z)) * W$ as stated in Section 2.3.

The first term in the last line is finite, uniformly in n , by Lemma 4.4. For the second term, note that by HY1 $\nabla_z f$ is continuous and, from (19) so is the family h_n . Furthermore, both $\nabla_z f(\cdot, \cdot, z)$ and $\nabla_z f(\cdot, \cdot, h_n(z))$ are dominated by $C(1 + |z|)$. Given the integrability properties of Z and the convergence of h_n to the identity function, dominated convergence yields the desired convergence result, from which the convergence of $(U_{u,t}^n, V_{u,t}^n)$ to $(D_u Y_t, D_u Z_t)$ for a.e. $t \in [0, T]$ follows.

(3) We prove the uniform boundedness of $\|(DU^n, DV^n)\|_{L^2(\mathbb{D}^{1,2})}^2$ in n .

The driver of BSDE (22) is linear. So applying Lemma 3.1, we obtain the following inequality for $n \in \mathbb{N}$

$$\|D_v U_u^n\|_{\mathcal{S}^{2p}}^{2p} + \|D_v V_u^n\|_{\mathcal{H}^{2p}}^{2p} \leq C \mathbb{E} \left[|D_v D_u \xi|^{2p\bar{q}^2} + \left(\int_0^T |A_{v,u,s}^n| ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}},$$

where the constant \bar{q} is related to the BMO martingale $\nabla_z f(\cdot, \Theta) * W$ according to Remark 2.7. Lemma 4.5 now yields

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq v, u \leq T} \left\{ \|D_v U_u^n\|_{\mathcal{S}^{2p}}^{2p} + \|D_v V_u^n\|_{\mathcal{H}^{2p}}^{2p} \right\} < \infty.$$

By (1) to (3) we can apply [Lemma 2.1](#) and deduce the Malliavin differentiability of (DY, DZ) . Arguments as the ones used in Theorem 8.4 of [1] show that $(D_v D_u Y, D_v D_u Z)$ is a solution to BSDE (18).

Uniqueness follows immediately from [Lemma 3.2](#).

To prove the representation $D_t D_u Y_t = D_u Z_t$, one only needs to recall that for each $n, u \leq t$ we have $D_t D_u Y_t^n = D_u Z_t^n$. Since both sides converge to their respective limiting processes the equality holds true in the limit. \square

5. Regularity in the time variable

With a view towards their numerical approximation, in this section we investigate regularity properties of the Malliavin derivatives of solutions of our qgFBSDE (1), (2).

In the following subsections the results presented are shown to hold mainly under assumptions HX1 and HY1. Several of these results can be proved under weaker conditions, namely by replacing HY1 with HY0. This is achieved by using a canonical argument of regularization followed by the application of Fatou's lemma. Because this type of reasoning is well known we state only [Theorem 5.6](#) under weaker assumptions.

5.1. Continuity and bounds

Lemma 5.1. *Under HX1 and HY1 let (X, Y, Z) be the solution processes of system (1), (2), and (DX, DY, DZ) their Malliavin derivatives. Then for $u, v \in [0, T]$ and $p \geq 1$ there exists a positive constant C_p such that*

$$\|D_v Y - D_u Y\|_{S^{2p}}^{2p} + \|D_v Z - D_u Z\|_{\mathcal{H}^{2p}}^{2p} \leq C_p |v - u|^p.$$

Proof. We use (10) to write for $u, v, t \in [0, T]$ a FBSDE for the difference $D_v Y_t - D_u Y_t$. For this we employ the comparison [Lemma 3.2](#), to obtain with $\xi = g(X_T)$ and for any $p \geq 1$

$$\begin{aligned} & \|D_v Y - D_u Y\|_{S^{2p}}^{2p} + \|D_v Z - D_u Z\|_{\mathcal{H}^{2p}}^{2p} \\ & \leq C \mathbb{E} \left[|D_v \xi - D_u \xi|^{2p\bar{q}^2} + \left(\int_0^T |(\nabla_x f)(s, \Theta_s)| |D_v X_s - D_u X_s| ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}} \\ & \leq C \mathbb{E} \left[|D_v X_T - D_u X_T|^{2p\bar{q}^2} \right. \\ & \quad \left. + \sup_{0 \leq t \leq T} |D_v X_t - D_u X_t|^{2p\bar{q}^2} \left(\int_0^T (1 + |Y_s| + |Z_s|^2) ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}} \\ & \leq C \left\{ \|D_v X_T - D_u X_T\|_{L^{2p\bar{q}^2}}^{2p} + \|D_v X - D_u X\|_{S^{4p\bar{q}^2}}^{2p} \right\} \leq C_p |v - u|^p, \end{aligned}$$

where \bar{q} corresponds to the BMO martingale $\nabla_z f(\Theta) * W$. The last line follows from a direct application of [Theorem 2.5](#). \square

Equipped with these moment estimates we are now able to state our first main result.

Theorem 5.2 (Time Continuity). *Assume HX1 and HY1. Then there exists a continuous version of $(u, t) \mapsto D_u Y_t$ in $\{(u, t) : 0 \leq u \leq t \leq T\}$. In particular there exists a continuous version of Z on $[0, T]$.*

Assume HX2 and HY2. Then there exists a continuous version of $(v, u, t) \mapsto D_v D_u Y_t$ for $0 \leq v \leq u \leq t \leq T$. In particular there is a continuous version of $(u, t) \mapsto D_u Z_t$ for $0 \leq u \leq t \leq T$.

Proof. To make the proof simpler we assume $m = d = 1$. Under HX1, the results of Section 2.5 imply the existence of continuous versions of $X, \nabla X, (\nabla X)^{-1}$ and $(u, t) \mapsto D_u X_t$ for $0 \leq u \leq t \leq T$.

A quick analysis of (9), combined with the knowledge that $(X, Y, Z) \in \mathcal{S}^{2p} \times \mathcal{S}^\infty \times \mathcal{H}^{2p}$ and $(\nabla X, \nabla Y, \nabla Z) \in \mathcal{S}^{2p} \times \mathcal{S}^{2p} \times \mathcal{H}^{2p}$ for all $p \geq 1$, allows one to conclude that a continuous version of ∇Y exists: the process is given by the sum of a Lebesgue and Itô integral with well-behaved integrands.

In Theorem 2.9 we established $D_u Y_t = \nabla Y_t (\nabla X_u)^{-1} \sigma(X_u)$, $0 \leq u \leq t \leq T$. Condition HX0 ensures the continuity of σ . Given that all terms in the representation of $D_u Y_t$ are continuous, we conclude that there is a continuous version of $(u, t) \mapsto (\nabla Y_t (\nabla X_u)^{-1} \sigma(X_u))$ for $0 \leq u \leq t \leq T$. This means that $(u, t) \mapsto D_u Y_t$ has a continuous version for $0 \leq u \leq t \leq T$.

By Theorem 2.9, Z is a version of $t \mapsto D_t Y_t$. Hence the continuity of a version of $(u, t) \mapsto D_u Y_t$ for $0 \leq u \leq t \leq T$ immediately implies that Z possesses a continuous version. This finishes the proof of the first statement.

For the second statement we argue in a different way.

The second Malliavin derivative of Y depends on three variables, $v, u, t \in [0, T]$. By using moment inequalities, we will show that $(v, u, t) \mapsto D_v D_u Y_t$ is continuous as a mapping to the space of continuous functions on $0 \leq v \leq u \leq t \leq T$ equipped with the sup norm. By well-known extensions of the Kolmogorov continuity criterion to normed vector spaces (see for example Theorem 1.4.1 in [10]) this will establish the desired continuity of $(v, u, t) \mapsto D_v D_u Y_t$ for $0 \leq v \leq u \leq t \leq T$. To verify the inequalities, for $0 \leq v \leq u \leq T$ and $0 \leq v' \leq u' \leq T$ we will have to estimate moments of

$$\sup_{0 \leq t \leq T} |D_v D_u Y_t - D_{v'} D_{u'} Y_t|^p.$$

In a first step, we separate the two parameters by estimating this quantity by a constant multiple of

$$\sup_{0 \leq t \leq T} |D_v D_u Y_t - D_{v'} D_u Y_t|^p + \sup_{0 \leq t \leq T} |D_{v'} D_u Y_t - D_{v'} D_{u'} Y_t|^p.$$

In what follows, for convenience we shall only give the estimation of the first summand, remarking that the second one may be treated in a very similar way. Fix $0 \leq v, v' \leq u \leq t \leq T$. Again using the comparison Lemma 3.2 with (18) specified to $D_v D_u Y_t - D_{v'} D_u Y_t$, we get for $p \geq 1$

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |D_v D_u Y_t - D_{v'} D_u Y_t|^{2p} \right] \\ & \leq C \left\{ \mathbb{E} \left[|D_v D_u \xi - D_{v'} D_u \xi|^{2p\bar{q}^2} + \left(\int_0^T \left[|D_v \Theta_s - D_{v'} \Theta_s| |Hf|(s, \Theta_s)| D_u \Theta_s| \right. \right. \right. \right. \\ & \quad \left. \left. \left. + |(\nabla_x f)(s, \Theta_s)| |D_v D_u X_s - D_{v'} D_u X_s| \right] ds \right)^{2p\bar{q}^2} \right] \right\} \\ & \leq C \left\{ \|D_v X_T - D_{v'} X_T\|_{L^{6p\bar{q}^2}}^{2p} + \|D_v D_u X - D_{v'} D_u X\|_{\mathcal{S}^{4p\bar{q}^2}}^{2p} \right. \\ & \quad \left. + \|D_v X - D_{v'} X\|_{\mathcal{S}^{4p\bar{q}^2}}^{2p} + \|D_v Y - D_{v'} Y\|_{\mathcal{S}^{4p\bar{q}^2}}^{2p} + \|D_v Z - D_{v'} Z\|_{\mathcal{H}^{8p\bar{q}^2}}^{4p} \right\} \\ & \leq C_p |v - v'|^p. \end{aligned}$$

The successive inequalities are justified in view of the growth conditions contained in the assumptions, Hölder's inequality, [Theorem 2.5](#) and [Lemma 5.1](#). Kolmogorov's continuity criterion for vector valued stochastic processes yields the existence of a continuous version of $(v, u, t) \mapsto D_v D_u Y_t$ for $0 \leq v \leq u \leq t \leq T$, and hence by restriction also of $(u, t) \mapsto D_u Z_t$ for $0 \leq u \leq t \leq T$. \square

Theorem 5.3 (Bounds). Assume that HX1 and HY1 hold. Then for all $p \geq 1$

$$\mathbb{E} \left[\sup_{0 \leq u \leq t \leq T} |D_u Y_t|^{2p} \right] < \infty.$$

In particular

$$\|Z\|_{\mathcal{S}^{2p}} < \infty. \quad (24)$$

Let HX2 and HY2 be satisfied. Then for all $p \geq 1$

$$\sup_{0 \leq u \leq T} \mathbb{E} \left[\sup_{0 \leq t \leq T} |D_u Z_t|^{2p} \right] < \infty.$$

Proof. As we have seen in [Theorem 5.2](#), a continuous version of $(u, t) \mapsto D_u Y_t$ is given by $\nabla_x Y_t (\nabla_x X_u)^{-1} \sigma(u, X_u)$. Hence we may estimate

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq u \leq t \leq T} |D_u Y_t|^{2p} \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |\nabla_x Y_t|^{2p} \sup_{0 \leq u \leq T} \{ |(\nabla_x X_u)^{-1} \sigma(u, X_u)|^{2p} \} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |\nabla_x Y_t|^{6p} \right]^{\frac{1}{3}} \mathbb{E} \left[\sup_{0 \leq u \leq T} |(\nabla_x X_u)^{-1}|^{6p} \right]^{\frac{1}{3}} \mathbb{E} \left[\sup_{0 \leq u \leq T} |\sigma(X_u)|^{6p} \right]^{\frac{1}{3}} \\ &< \infty. \end{aligned}$$

The last line follows from the fact that ∇Y , $(\nabla X)^{-1}$ and X all belong to \mathcal{S}^{2p} for all $p \geq 1$ (see [\(4\)](#), [\(7\)](#) and [Theorem 2.8](#)). This concludes the first part of the proof. The second claim follows as a special case of the first by identifying u and t .

For the third statement, note that the proof of [Theorem 4.1](#) (see also the proof of [Theorem A.1](#)) yields

$$\sup_{0 \leq v, u \leq T} \mathbb{E} \left[\sup_{0 \leq t \leq T} |D_v D_u Y_t|^{2p} \right] < \infty, \quad p \geq 1.$$

By the continuity result of [Theorem 5.2](#) we may choose $u = t$ to obtain

$$\sup_{0 \leq u \leq T} \mathbb{E} \left[\sup_{0 \leq t \leq T} |D_u Z_t|^{2p} \right] < \infty, \quad p \geq 1. \quad \square$$

5.2. A path regularity theorem

In the previous subsection we deduced the continuity property of Z and estimated moments of its supremum over the interval $[0, T]$. Here, we aim at providing a Kolmogorov continuity type estimate for Z . The inequality we will obtain will imply an improvement of the well-known path regularity result stated in [\[12,13\]](#).

Let Π be the collection of all partitions of the interval $[0, T]$ by finite families of real numbers. Particular partitions will be denoted by $\pi = \{t_i : 0 = t_0 < \dots < t_N = T\}$ with $N \in \mathbb{N}$. We define the mesh size of partition π as $|\pi| = \max_{0 \leq i \leq N} |t_{i+1} - t_i|$.

For reference purposes and before approaching the path regularity theorem we recall an elementary inequality: for real numbers a_i , $1 \leq i \leq n$, and $p \geq 1$ we have

$$\sum_{i=1}^n |a_i|^p \leq \left(\sum_{i=1}^n |a_i| \right)^p. \quad (25)$$

We start by stating an auxiliary lemma.

Lemma 5.4. Assume $HX0$ and $HY0$. Then for the solutions of BSDE (1), (2) and for any $p \geq 2$ there exists a pair of constants A_p, C_p depending on T, M and p such that

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u - Y_s|^p \right] \leq C_p \left\{ A_p |t - s|^p + \mathbb{E} \left[\left(\int_s^t |Z_v|^2 dv \right)^p + \left(\int_s^t |Z_v|^2 dv \right)^{p/2} \right] \right\}. \quad (26)$$

Proof. First estimate increments of Y by the sum of a Lebesgue and Itô integral provided by (2), maximize in $s \leq u \leq t$, and apply Doob's and Burkholder–Davis–Gundy's inequalities to the martingale part to obtain for $p \geq 2$

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u - Y_s|^p \right] \leq C_p \mathbb{E} \left[\left(\int_s^t |f(v, X_v, Y_v, Z_v)| dv \right)^p + \left(\int_s^t |Z_v|^2 dv \right)^{\frac{p}{2}} \right].$$

Next use the growth condition valid for f , i.e. $|f(\cdot, \cdot, y, z)| \leq M(1 + |y| + |z|^2)$ together with the fact that Y is bounded, to obtain the claimed result. \square

Let us now state our path regularity theorem.

Theorem 5.5 (Path Regularity). Under $HX1$ and $HY1$, the FBSDE system (1), (2) has a unique solution $(X, Y, Z) \in \mathcal{S}^{2p} \times \mathcal{S}^\infty \times \mathcal{H}^{2p}$ for all $p \geq 1$. Moreover, the following holds true:

(i) For $p \geq 2$ there exists a constant $C_p > 0$ such that for $0 \leq s \leq t \leq T$ we have

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u - Y_s|^p \right] \leq C_p |t - s|^{\frac{p}{2}}.$$

(ii) For all $p \geq 1$ there exists a constant $C_p > 0$ such that for any partition π of $[0, T]$ with N points and mesh size $|\pi|$

$$\sum_{i=0}^{N-1} \mathbb{E} \left[\left(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 dt \right)^p \right] \leq C_p |\pi|^p.$$

Under $HX2$ and $HY2$, we further have:

(iii) For all $p \geq 2$ there exists a constant $C_p > 0$ such that for $0 \leq s \leq t \leq T$

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |Z_u - Z_s|^p \right] \leq C_p |t - s|^{\frac{p}{2}}.$$

In particular, the process Z has a continuous modification.

Proof. Part (i): Under the hypotheses we can make use of Theorem 5.3. In fact, combining (24) with (26) we get

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u - Y_s|^p \right] &\leq C \left\{ |t - s|^p + \mathbb{E} \left[|t - s|^p \sup_{s \leq u \leq t} |Z_u|^{2p} + |t - s|^{\frac{p}{2}} \sup_{s \leq u \leq t} |Z_u|^p \right] \right\} \\ &\leq C_p \left\{ |t - s|^p + |t - s|^{\frac{p}{2}} \right\}. \end{aligned}$$

The result follows.

Part (ii): **Theorem 5.3** states that $Z \in \mathcal{S}^{2p}$. Therefore we are able to write, using Jensen's inequality

$$\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 dt\right)^p\right] \leq |\pi|^{p-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|Z_t - Z_{t_i}|^{2p}] dt. \quad (27)$$

In view of **Theorem 5.2** and the subsequent representation formula for Z in terms the Malliavin derivatives of Y (see (11)), we find an alternative way to express the difference $Z_t - Z_{t_i}$ for $t \in [t_i, t_{i+1}]$ by writing

$$Z_t - Z_{t_i} = \nabla Y_t (\nabla X_t)^{-1} \sigma(X_t) - \nabla Y_{t_i} (\nabla X_{t_i})^{-1} \sigma(X_{t_i}) = I_1 + I_2 + I_3, \quad (28)$$

where $I_1 = (\nabla Y_t - \nabla Y_{t_i})(\nabla X_t)^{-1} \sigma(X_t)$, $I_2 = \nabla Y_{t_i}((\nabla X_t)^{-1} - (\nabla X_{t_i})^{-1}) \sigma(X_t)$ and $I_3 = \nabla Y_{t_i} (\nabla X_{t_i})^{-1} (\sigma(X_t) - \sigma(X_{t_i}))$.

Estimates for I_2 and I_3 are easy to produce since they rely mainly on $\|\nabla Y\|_{\mathcal{S}^{2p}} < \infty$ and the results presented in Section 2.5. We give details for I_2 and hints on how to deal with I_3 , remarking that its treatment is very similar. Hölder's inequality combined with the growth condition of σ produce

$$\begin{aligned} \mathbb{E}[|I_2|^{2p}] &\leq C \mathbb{E}\left[\sup_{0 \leq u \leq T} |\nabla_x Y_u|^{6p}\right]^{\frac{1}{3}} \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} |(\nabla X_t)^{-1} - (\nabla X_{t_i})^{-1}|^{6p}\right]^{\frac{1}{3}} \\ &\leq C |\pi|^{3p \frac{1}{3}} = C |\pi|^p. \end{aligned} \quad (29)$$

For the last line we use (4), (8) and $\|\nabla Y\|_{\mathcal{S}^{2p}} < \infty$. For I_3 , the method is similar: instead of (4) and (8) we have to use (5) and (7).

We next estimate I_1 . Using Fubini's theorem and Hölder's inequality we get

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \mathbb{E}[|I_1|^{2p}] dt &= \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |I_1|^{2p} dt\right] \\ &\leq \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |(\nabla X_t)^{-1}|^{4p} dt\right]^{\frac{1}{2}} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |\sigma(X_t)|^{4p} dt\right]^{\frac{1}{2}} \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p}\right]. \end{aligned}$$

We can simplify the integral terms by estimating the integrands by their suprema over the intervals. Using the linear growth condition on σ combined with (4), (7), we show in this way that the first two expectations on the right-hand side are bounded by $C|\pi|^{1/2}$ each. Applying an appropriate version of (27), and using the previous inequalities, we infer

$$|\pi|^{p-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|I_1|^p] dt \leq C |\pi|^p \sum_{i=0}^{N-1} \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p}\right].$$

It remains to estimate $\nabla Y_t - \nabla Y_{t_i}$ for $t \in [t_i, t_{i+1}]$ using the BSDE (9). For $p \geq 1$, the inequalities of Doob and Burkholder–Davis–Gundy combine with HX1 and HY1 in the same fashion as in part (i) to yield for $\Theta = (X, Y, Z)$ and $\nabla \Theta = (\nabla X, \nabla Y, \nabla Z)$

$$\sum_{i=0}^{N-1} \mathbb{E}\left[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p}\right]$$

$$\begin{aligned} &\leq C \sum_{i=0}^{N-1} \mathbb{E} \left[\left(\int_{t_i}^{t_{i+1}} |(\nabla f)(s, \Theta_s), \nabla \Theta_s| ds \right)^{2p} + \left(\int_{t_i}^{t_{i+1}} |\nabla Z_s|^2 ds \right)^p \right] \\ &\leq C \mathbb{E} \left[\left(\int_0^T |(\nabla f)(s, \Theta_s), \nabla \Theta_s| ds \right)^{2p} + \left(\int_0^T |\nabla Z_s|^2 ds \right)^p \right]. \end{aligned}$$

For the last line we interchange summation and expectation and apply (25). We now use the growth condition of **HY1** combined with the fact that $X, Y, Z, \nabla X, \nabla Y \in \mathcal{S}^{2p}$ and $\nabla Z \in \mathcal{H}^{2p}$. Therefore

$$\sum_{i=0}^{N-1} \mathbb{E} \left[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p} \right] < \infty,$$

which obviously implies

$$|\pi|^{p-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|I_1|^p] dt \leq C |\pi|^p.$$

Finally we inject (29) and the above inequality into (27) (according to (28)), to obtain the second assertion of the theorem:

$$\begin{aligned} \sum_{i=0}^{N-1} \mathbb{E} \left[\left(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 dt \right)^p \right] &\leq C |\pi|^{p-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[|I_1|^{2p} + |I_2|^{2p} + |I_3|^{2p}] dt \\ &\leq C |\pi|^{p-1} (|\pi| + 2|\pi|^p) \leq C |\pi|^p. \end{aligned}$$

Part (iii): Theorem 5.2 states that the map $t \mapsto D_t Y_t$ is a continuous version of $t \mapsto Z_t$, for $t \in [0, T]$. Hence we are able to express for $s, t \in [0, T]$ the difference $Z_t - Z_s$ by Malliavin derivatives of Y , and its moments for $p \geq 2$ by

$$\mathbb{E}[|Z_t - Z_s|^p] \leq C(\mathbb{E}[|D_t Y_t - D_s Y_t|^p] + \mathbb{E}[|D_s Y_t - D_s Y_s|^p]), \quad \text{with } s \leq t.$$

We estimate both expressions on the right-hand side separately. The arguments we use are similar to the ones in *Part (ii)*.

From Lemma 5.1 we have that $\mathbb{E}[|D_t Y_t - D_s Y_t|^p] \leq C|t - s|^{\frac{p}{2}}$. For the other term, a simple calculation using BSDE (10) yields for $t, s \in [0, T], 0 \leq s \leq t \leq T$

$$D_s Y_t - D_s Y_s = \int_s^t [\langle \nabla f(\Theta_u), D_s \Theta_u \rangle] du - \int_s^t D_s Z_u dW_u.$$

By Doob's and Burkholder–Davis–Gundy's inequalities we have for $p \geq 2$

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq u \leq t} |D_s Y_u - D_s Y_s|^p \right] &\leq C \mathbb{E} \left[\left(\int_s^t [(1 + |Y| + |Z|^2)|D_s X_u| + |D_s Y_u| \right. \right. \\ &\quad \left. \left. + (1 + |Z_u|)|D_s Z_u|] du \right)^p + \left(\int_s^t |D_s Z_u|^2 du \right)^{\frac{p}{2}} \right] \\ &\leq C \{ |t - s|^p + |t - s|^{\frac{p}{2}} \}. \end{aligned}$$

This last line follows, because all the integrand processes belong to \mathcal{S}^p for all $p \geq 2$ (see Theorem 5.3). Combining the two estimates above we have $\mathbb{E}[|Z_t - Z_s|^p] \leq C|t - s|^{\frac{p}{2}}$ as intended. Kolmogorov's continuity criterion yields the continuity statement. \square

5.3. The path regularity theorem for qgBSDE

Now let π be a partition of the interval $[0, T]$ with N points and mesh size $|\pi|$. We define a set of random variables

$$\bar{Z}_{t_i}^\pi = \frac{1}{t_{i+1} - t_i} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} Z_s ds \middle| \mathcal{F}_{t_i} \right], \quad \text{for all partition points } t_i, \quad 0 \leq i \leq N-1, \quad (30)$$

where Z is the control process in the solution of qgFBSDE (1), (2) under HX0 and HY0. It is not difficult to show that $\bar{Z}_{t_i}^\pi$ is the best \mathcal{F}_{t_i} -adapted $\mathcal{H}^2([t_i, t_{i+1}])$ approximation of Z , i.e.

$$\mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}^\pi|^2 ds \right] = \inf_{Z_i \in L^2(\Omega, \mathcal{F}_{t_i})} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s - Z_i|^2 ds \right].$$

Let now $\bar{Z}_t^\pi = \bar{Z}_{t_i}^\pi$ for $t \in [t_i, t_{i+1}]$, $0 \leq i \leq N-1$. It is equally easy to see that \bar{Z}^π converges to Z in \mathcal{H}^2 as $|\pi|$ vanishes: since Z is adapted there exists an adapted family of processes Z^π indexed by our partition such that $Z_t^\pi = Z_{t_i}$ for $t \in [t_i, t_{i+1})$ and that Z^π converges to Z in \mathcal{H}^2 as $|\pi|$ goes to zero. Since $\{\bar{Z}^\pi\}$ is the best \mathcal{H}^2 -approximation of Z , we obtain

$$\|Z - \bar{Z}^\pi\|_{\mathcal{H}^2} \leq \|Z - Z^\pi\|_{\mathcal{H}^2} \rightarrow 0, \quad \text{as } |\pi| \rightarrow 0.$$

As an immediate corollary of (ii) in the previous theorem we get the extension to the setting of drivers with quadratic growth of the famous Theorem 3.4.3 in [12]. Let $p = 1$ in Theorem 5.5. Then

Theorem 5.6. Assume HX1 and HY0. Assume further that condition (3) holds and that g satisfies a standard Lipschitz condition with Lipschitz constant M . Then there exists a positive constant C such that for any partition $\pi = \{t_0 < \dots < t_N\}$ of the interval $[0, T]$ with mesh size $|\pi|$ we have

$$\max_{0 \leq i \leq N-1} \left\{ \sup_{t \in [t_i, t_{i+1})} \mathbb{E} \left[|Y_t - Y_{t_i}|^2 \right] \right\} + \sum_{i=0}^{N-1} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}^\pi|^2 ds \right] \leq C|\pi|.$$

Proof. Let $(f^\varepsilon)_{\varepsilon>0}$ and $(g^\varepsilon)_{\varepsilon>0}$ be two families of C_0^∞ functions obtained by canonically regularizing f and g respectively, and such that

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d} |f^\varepsilon(t, x, y, z) - f(t, x, y, z)| + \sup_{x \in \mathbb{R}^m} |g^\varepsilon(x) - g(x)| \right\} = 0.$$

Since f satisfies condition HY0 and g is uniformly Lipschitz, for any $\varepsilon > 0$ both g^ε and f^ε satisfy condition HY1. Let us denote $(X, Y^\varepsilon, Z^\varepsilon)$ the solution of (1) and

$$Y_t^\varepsilon = g^\varepsilon(X_T) + \int_t^T f^\varepsilon(s, X_s, Y_s^\varepsilon, Z_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dW_s, \quad t \in [0, T], \quad \varepsilon > 0.$$

Then all the results of Section 5 proved so far hold for the pair $(Y^\varepsilon, Z^\varepsilon)_{\varepsilon>0}$. Using Lemma 3.2 we can conclude that for any $p \geq 1$

$$\lim_{\varepsilon \rightarrow 0} \left\{ \|Y^\varepsilon - Y\|_{\mathcal{S}^{2p}} + \|Z^\varepsilon - Z\|_{\mathcal{H}^{2p}} \right\} = 0.$$

We next apply Theorem 5.5. After a careful inspection of the arguments of its proof, we find a positive constant C independent of $\varepsilon > 0$ such that for any partition $\pi = \{t_0 < \dots < t_N\}$ of the

interval $[0, T]$ with mesh size $|\pi|$ we have

$$\max_{0 \leq i \leq N-1} \left\{ \sup_{t \in [t_i, t_{i+1})} \mathbb{E} \left[|Y_t^\varepsilon - Y_{t_i}^\varepsilon|^2 \right] \right\} + \sum_{i=0}^{N-1} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s^\varepsilon - \bar{Z}_{t_i}^{\varepsilon, \pi}|^2 ds \right] \leq C|\pi|,$$

with the set $\{\bar{Z}_{t_i}^{\varepsilon, \pi}\}_{t_i \in \pi}$ given as in (30) for the process $(Z_t^\varepsilon)_{t \in [0, T]}$. We finally apply Fatou's lemma to obtain

$$\sum_{i=0}^{N-1} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}^\pi|^2 ds \right] \leq \liminf_{\varepsilon \rightarrow 0} \sum_{i=0}^{N-1} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} |Z_s^\varepsilon - \bar{Z}_{t_i}^{\varepsilon, \pi}|^2 ds \right] \leq C|\pi|.$$

A similar argument holds for the difference $(Y_t^\varepsilon - Y_{t_i}^\varepsilon)$, $t_i \in \pi$ and $t \in [t_i, t_{i+1})$.

The result now follows. \square

6. Numerics for qgFBSDE—A truncation procedure

A common method to deal with nonlinearities or unbounded functions consists in truncating them. In our BSDE (2), the driver has a quadratic nonlinearity in z . Our truncation of the nonlinear driver will relate BSDE with drivers of quadratic growth BSDE with globally Lipschitz drivers. For this type of BSDE numerical schemes are readily available (see [2,5] and the references therein), and the error committed in the numerical approximation for BSDE with Lipschitz driver is well known. So to fully analyze the error related to successive approximations in the case of BSDE with drivers of quadratic growth, it only remains to provide an estimate for the error arising from the truncation. This is what we propose to do in this section.

Emphasizing once more the point made in the introduction, we remark that the convergence rate for numerical schemes of truncated qgFBSDE is well known and produces an exponential dependence on the truncation level. Despite this fact, the result presented in Theorem 6.2 implies that one can obtain a high convergence order for the truncation procedure. This modestly mitigates the exponential dependence of the convergence order of the scheme on the truncation level.

For our qgFBSDE system (1), (2) we assume that HX1 and HY1 hold. In this section the diffusion process X appearing in the BSDE's terminal condition and driver plays a secondary role, especially in the calculations we will be presenting.

To truncate the driver of quadratic growth, we use the already familiar sequence $\{h_n\}_{n \in \mathbb{N}}$ defined in (19). To justify that this sequence indeed does the job, we will need (24) to make our calculations work. At first we have to justify that the truncated FBSDE obtained in this way satisfies HX1 and HY1.

Recalling the driver of BSDE (2), we define a family of functions $f_n : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $(\omega, t, x, y, z) \mapsto f(\omega, t, x, y, h_n(z))$ and with it, the family of truncated FBSDE

$$Y_t^n = \xi + \int_t^T f(s, X_s, Y_s^n, h_n(Z_s^n)) ds - \int_t^T Z_s^n dW_s, \quad (31)$$

with $\xi = g(X_T)$. The solution process of (2) is denoted by (Y, Z) and the solution process of its truncated counterpart (31) by (Y^n, Z^n) . Furthermore, we recall that in (19) $(h_n)_{n \in \mathbb{N}}$ was defined as a sequence of C^1 functions, and that by Theorem 2.6 we have

$$\max \left\{ \sup_{n \in \mathbb{N}} \|Z^n * W\|_{BMO}, \|Z * W\|_{BMO} \right\} \leq \frac{4+6M^2T}{3M^2} \exp \left\{ 6M \|\xi\|_{L^\infty} + MT \right\} < \infty. \quad (32)$$

This means that the martingales of the sequence $(Z^n * W)_{n \in \mathbb{N}}$ satisfy the reverse Hölder inequality with an exponent \bar{r} independent of n (see Section 2.3, and also Remark 4.3).

Remark 6.1. If (2) satisfies HX1 and HY1, by inspection of the hypotheses it is easy to see that family (31) also satisfies HX1 and HY1 uniformly in n . This means that the results on differentiability in Section 2.6 and on continuity and bounds in Section 5.1 are available for the truncated BSDE (31) as well.

The proof of our result on the truncation error relies on Markov's inequality and (24). The convergence rate will depend on a parameter which arises from the inverse Hölder inequality and is related to (32).

Theorem 6.2. Assume that HX1 and HY1 are satisfied, and let (Y, Z) and (Y^n, Z^n) be solutions of (2) and (31) respectively. Then for any $p \geq 1$ and $\beta \geq 1$ there exist positive finite constants C_p and D_β such that for $n \in \mathbb{N}$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^n - Y_t|^{2p} \right] + \mathbb{E} \left[\left(\int_0^T |Z_s^n - Z_s|^2 ds \right)^p \right] \leq C_p D_\beta n^{-\frac{\beta}{2\bar{q}}}.$$

The constant \bar{q} is the Hölder conjugate of $\bar{r} \in (1, \infty)$ which is related to the estimate (32) according to Remark 2.7. The constant D_β is given by $D_\beta = \left(\sup_{n \in \mathbb{N}} \|Z^n\|_{\mathcal{S}^{2\beta\bar{q}}(\mathbb{P})} \right)^{\frac{\beta}{2\bar{q}}}$. The constant C_p is independent of $\beta \geq 1$.

Proof. Let $n \in \mathbb{N}$ and $t \in [0, T]$. As usual we have to rely on an a priori estimate for the difference of original and truncated BSDE. To this end we use the notation, methods and arguments of the proofs of Lemmas 3.1 and 3.2 (see Section 3), without repeating all the details. For

$$b_t^n = \frac{f_n(t, X_t, Y_t^n, Z_t^n) - f(t, X_t, Y_t^n, Z_t)}{|Z_t^n - Z_t|^2} (Z_t^n - Z_t) \mathbb{1}_{\{Z_t^n \neq Z_t\}}, \quad t \in [0, T], n \in \mathbb{N}.$$

The associated equivalent measure obtained after measure change by subtracting the drift related to b^n will be denoted by \mathbb{Q}^{b^n} . The superscript will be omitted for convenience. Using Hölder's inequality and for some positive real constant C we obtain,

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_t - Y_t^n|^{2p} + \left(\int_0^T |Z_s - Z_s^n|^2 ds \right)^p \right] \\ & \leq D \mathbb{E}^{\mathbb{Q}} \left[\sup_{t \in [0, T]} |Y_t - Y_t^n|^{2p\bar{q}} + \left(\int_0^T |Z_s - Z_s^n|^2 ds \right)^{p\bar{q}} \right]^{\frac{1}{\bar{q}}}, \end{aligned} \quad (33)$$

with $D = \sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{P}} [\mathcal{E}(b_n * W)^{\bar{r}}]^{\frac{1}{\bar{r}}} < \infty$. That D is finite follows from (32) combined with part (2) of Lemma 2.2.

We continue with the estimation of (33). Following the proof of Lemma 3.1 or Lemma 3.2 (see (16)), there exists a positive constant C such that

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} \left[\sup_{0 \leq t \leq T} |Y_t - Y_t^n|^{2p\bar{q}} + \left(\int_0^T |Z_s - Z_s^n|^2 ds \right)^{p\bar{q}} \right]^{\frac{1}{\bar{q}}} \\ & \leq C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, h_n(Z_s^n))| ds \right)^{2p\bar{q}} \right]^{\frac{1}{\bar{q}}} \end{aligned}$$

$$\begin{aligned}
&\leq C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T M(1 + |Z_s^n| + |h_n(Z_s^n)|) |Z_s^n - h_n(Z_s^n)| \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{\bar{q}}} \\
&\leq C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |M(1 + |Z_s^n| + |h_n(Z_s^n)|)|^2 \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{2\bar{q}}} \\
&\quad \times \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |Z_s^n - h_n(Z_s^n)|^2 \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{2\bar{q}}} \\
&\leq C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |Z_s^n - h_n(Z_s^n)|^2 \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{2\bar{q}}}, \tag{34}
\end{aligned}$$

where we made use of the growth assumption on f stated in HY1, Hölder's inequality and (16).

A closer look at the properties of h_n reveals that for any $n \in \mathbb{N}$ and $s \in [0, T]$ we have

$$|Z_s^n - h_n(Z_s^n)|^2 \leq 4|Z_s^n|^2 \mathbb{1}_{\{|Z_s^n| > n\}}.$$

In view of this inequality, an explicit convergence rate can be obtained if the term $\mathbb{1}_{\{|Z_s^n| > n\}}$ is explored. Because (24) holds we can use Markov's inequality for this purpose.

As pointed out in Remark 6.1, the validity of HX1 and HY1 for the family of drivers used entitles us to employ the crucial (24) of Theorem 5.3. We now develop (34) using sequentially Hölder's inequality, Jensen's inequality and Fubini's theorem, to obtain

$$\begin{aligned}
C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |Z_s^n - h_n(Z_s^n)|^2 \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{2\bar{q}}} &\leq C \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T |Z_s^n|^4 \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{4\bar{q}}} \\
&\quad \times \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^T \mathbb{1}_{\{|Z_s^n| > n\}} \, ds \right)^{2p\bar{q}} \right]^{\frac{1}{4\bar{q}}} \leq C \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \mathbb{1}_{\{|Z_s^n| > n\}} \, ds \right]^{\frac{1}{4\bar{q}}} \\
&\leq C \left(\int_0^T \mathbb{E}^{\mathbb{Q}} [\mathbb{1}_{\{|Z_s^n| > n\}}] \, ds \right)^{\frac{1}{4\bar{q}}} = C \left(\int_0^T \mathbb{Q}[\{|Z_s^n| > n\}] \, ds \right)^{\frac{1}{4\bar{q}}}.
\end{aligned}$$

Applying Markov's inequality we obtain for some $\beta \geq 1$

$$\begin{aligned}
C \left(\int_0^T \frac{1}{n^{2\beta}} \mathbb{E}^{\mathbb{Q}} [|Z_s^n|^{2\beta}] \, ds \right)^{\frac{1}{4\bar{q}}} &= C n^{-\frac{\beta}{2\bar{q}}} \mathbb{E}^{\mathbb{Q}} \left[\int_0^T |Z_s^n|^{2\beta} \, ds \right]^{\frac{1}{4\bar{q}}} \\
&\leq C D \mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |Z_t|^{2\beta\bar{q}} \right]^{\frac{1}{4\bar{q}^2}} n^{-\frac{\beta}{2\bar{q}}},
\end{aligned}$$

with D as in inequality (33). We emphasize that the constant C which varies from line to line depends on p , T , \bar{r} and \bar{q} , but *not* on n or β .

By construction, it is clear that

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} |Y_t - Y_t^n|^2 p + \left(\int_0^T |Z_s - Z_s^n|^2 \, ds \right)^p \right] \leq C \left(\sup_{n \in \mathbb{N}} \|Z^n\|_{\mathcal{S}^{2\beta\bar{q}}(\mathbb{P})} \right)^{\frac{\beta}{2\bar{q}}} n^{-\frac{\beta}{2\bar{q}}},$$

with a positive constant C independent of β and n .

We finish this proof with an argument establishing the finiteness of $\sup_{n \in \mathbb{N}} \|Z^n\|_{\mathcal{S}^\gamma}$ for $\gamma > 2$. Having in mind Remark 6.1 we can apply, for every $n \in \mathbb{N}$, Theorem 2.8 to BSDE (31). We obtain that for each n that the pair (Y^n, Z^n) is differentiable with derivatives given by $(\nabla Y^n, \nabla Z^n)$. The derivatives satisfy BSDE (9) with the driver f replaced by the corresponding driver f_n (see BSDE (31)).

Given the properties of the sequence $(h_n)_{n \in \mathbb{N}}$ and inequality (32), we can apply Lemma 3.1 to the BSDE for $(\nabla Y^n, \nabla Z^n)$ and easily obtain that for any $\gamma \geq 2$, $\sup_{n \in \mathbb{N}} \|\nabla Y^n\|_{\mathcal{S}^\gamma} < \infty$.

With arguments similar to those used to prove (24), it follows that for any $\gamma \geq 2$ we have $\sup_{n \in \mathbb{N}} \|Z^n\|_{\mathcal{S}^\gamma} < \infty$. \square

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Appendix

In this Appendix we give the technical details left out in Section 4 in the proof of second order Malliavin differentiability of the solution processes of a BSDE the driver of which satisfies Lipschitz conditions.

The techniques we will use are not new. They are based on a Picard iteration argument. It does not only give existence and uniqueness of solutions but also allows establishing Malliavin differentiability in each step for the respective approximation of the solution. By means of a contraction argument in a suitable Sobolev norm, Malliavin smoothness is carried over to the solution in the limit. In contrast to previous applications, here the scheme deals with an equation that already has a Malliavin derivative as its solution.

We start with canonical coefficients that are given by an \mathcal{F}_T -measurable random variable ξ and a measurable function $f : \Omega \times [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that

$$f(\cdot, s, u, y, z) = a_{u,s}(\cdot) + b_s(\cdot)y + \langle c_s(\cdot), z \rangle.$$

For the remainder we omit the dependence of the coefficients on $\omega \in \Omega$. The coefficient functions defining this driver will be supposed to satisfy the following assumptions.

- (A1) $b : \Omega \times [0, T] \rightarrow \mathbb{R}$ and $c : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ are measurable (\mathcal{F}_t) -adapted processes bounded by a constant $M > 0$.
 $a : \Omega \times [0, T] \times [0, T] \rightarrow \mathbb{R}$ satisfies $\sup_{0 \leq u \leq T} \|a_{u,\cdot}\|_{\mathcal{S}^{2p}} < \infty$ for all $p \geq 1$. For each fixed $u \in [0, T]$ the process $(a_{u,t})_{t \in [0, T]}$ is progressively measurable.
- (A2) ξ is a Malliavin differentiable, \mathcal{F}_T -measurable bounded random variable with Malliavin derivative given by $D\xi$ satisfying

$$\sup_{0 \leq u \leq T} \|D_u \xi\|_{L^{2p}} < \infty, \quad \text{for all } p \geq 1.$$

- (A3) $a_{u,t}, b_t, c_t$ are Malliavin differentiable for all $u \in [0, T]$ and $t \in [0, T]$. Measurable versions of their Malliavin derivatives are respectively given by $D_v a_{u,t}, D_v b_t$ and $D_v c_t$ for $v \in [0, T]$ such that for all $p \geq 1$

$$\sup_{0 \leq v, u \leq T} \mathbb{E} \left[\left(\int_0^T [|D_v a_{u,s}|^2 + |D_v b_s|^2 + |D_v c_s|^2] ds \right)^p \right] < \infty.$$

- (A4) For all $u \in [0, T]$, $D_u \xi$ is Malliavin differentiable, with second order derivative given by $D_v D_u \xi$, $v, u \in [0, T]$, and satisfying $\sup_{0 \leq u, v \leq T} \|D_v D_u \xi\|_{L^p} < \infty$ for all $p \geq 1$.

Under these assumptions we consider the following backward stochastic differential equation

$$\begin{aligned} U_{u,t} &= 0, & V_{u,t} &= 0, & t &\in [0, u), \\ U_{u,t} &= D_u \xi - \int_t^T V_{u,s} dW_s + \int_t^T f(s, u, U_{u,s}, V_{u,s}) ds, & t &\in [u, T]. \end{aligned} \quad (35)$$

Theorem A.1. Under (A1) and (A2), the BSDE (35) has a unique solution (U, V) in $S^{2p} \times \mathcal{H}^{2p}$ for $p \geq 1$. Furthermore, if (A3) and (A4) hold, then (U, V) is Malliavin differentiable and a version of $\{(D_v U_{u,t}, D_v V_{u,t}); 0 \leq v \leq u \leq t \leq T\}$ satisfies for any $0 \leq v \leq u \leq t \leq T$

$$D_v U_{u,t} = D_v D_u \xi - \int_t^T D_v V_{u,s} dW_s + \int_t^T \left[(D_v f)(s, u, U_{u,s}, U_{u,s}) + \langle \nabla f(s, u, U_{u,s}, U_{u,s}), (D_v U_{u,s}, D_v U_{u,s}) \rangle \right] ds. \quad (36)$$

A version of $\{V_{u,t}; 0 \leq u \leq t \leq T\}$ is given by $\{D_t U_{u,t}; 0 \leq u \leq t \leq T\}$.

Proof. For the sake of notational simplicity and clarity, we provide a proof for the case $d = 1$. This proof splits into two steps. In the first one, we are concerned with existence and uniqueness of solutions for (35) using a Picard iteration. In the second step we prove Malliavin differentiability. To this end, we show that the sequence arising in the Picard scheme is in fact Malliavin differentiable, and by contraction that its limit must be the Malliavin derivative of the solution process constructed in the first part.

Part (i): To simplify notation we refer to $D_u \xi$ as ξ_u if there is no ambiguity.

We have a “standard” BSDE with Lipschitz continuous driver and a smooth terminal condition. The usual arguments for existence and uniqueness in this setting are well known. We recall the Picard iteration argument of the proof of Proposition 5.3 in [6]. Let $(U_{u,t}^0, V_{u,t}^0) = (0, 0)$ and for $k \geq 0$ define recursively the pair $(U_{u,t}^{k+1}, V_{u,t}^{k+1})$ as the solution of

$$U_{u,t}^{k+1} = D_u \xi - \int_t^T V_{u,s}^{k+1} dW_s + \int_t^T f(s, u, U_{u,s}^k, V_{u,s}^k) ds, \quad \text{for } 0 \leq u \leq t \leq T.$$

Under (A1) and (A2) the iteration scheme is well defined and the following moment estimates hold for all $k \in \mathbb{N}$ and $p \geq 1$ (see Proposition 2.1 in [6]):

$$\sup_{0 \leq u \leq T} \left\{ \|U_u^k\|_{S^{2p}}^{2p} + \|V_u^k\|_{\mathcal{H}^{2p}}^{2p} \right\} \leq C \sup_{0 \leq u \leq T} \left\{ \|D_u \xi\|_{L^{2p}}^{2p} + \|a_{u,\cdot}\|_{\mathcal{H}^{2p}}^{2p} \right\} < \infty.$$

Along the classical lines of the argument⁴ of Corollary 2.1 in [6] we obtain: $(U^k, V^k) \rightarrow (U, V) d\mathbb{P} \otimes dt \otimes du$ a.e. as well as

$$\lim_{k \rightarrow \infty} \sup_{0 \leq u \leq T} \left\{ \|U_u^{k+1} - U_u\|_{S^{2p}} + \|V_u^{k+1} - V_u\|_{\mathcal{H}^{2p}} \right\} = 0, \quad \text{for all } p \geq 1.$$

Part (ii): In what follows we prove that the sequence $(U^k, V^k)_{k \in \mathbb{N}}$ is Malliavin differentiable and its Malliavin derivatives converge to a version of the Malliavin derivative of (U, V) , the arguments we use are close to the ones in [5]. The proof is done recursively, starting with the initial step. From (A3) and (A4), the differentiability of ξ_u and $a_{u,t}$ implies that for all $0 \leq u \leq t \leq T$ the process $\mathbb{E}[\xi_u + \int_t^T f(s, u, 0, 0) ds | \mathcal{F}_t] \in \mathbb{D}^{1,2}$ and hence

$$\mathbb{E} \left[\xi_u + \int_t^T f(s, u, 0, 0) ds | \mathcal{F}_t \right] = U_{u,t}^1 \in \mathbb{D}^{1,2}.$$

⁴ Applying the Itô formula to $e^{\beta t} (U_{u,t}^{k+1} - U_{u,t}^k)^2$, $t \in [0, T]$, one proves in the usual fashion norm contraction of the sequence through the a priori estimates. As this argument is well known we omit it.

Since $\xi + \int_t^T f(s, u, 0, 0)ds - U_{u,t}^1 = \int_t^T V_{u,s}^1 dW_s$, Lemma 5.1 of [6] implies $V_{u,t}^1 \in \mathbb{D}^{1,2}$. For the recursive step, we next show that if $(U_{u,t}^k, V_{u,t}^k) \in \mathbb{D}^{1,2}$, then also $(U_{u,t}^{k+1}, V_{u,t}^{k+1}) \in \mathbb{D}^{1,2}$. Assume that $(U_{u,t}^k, V_{u,t}^k) \in \mathbb{D}^{1,2}$. Since $b, c \in \mathbb{D}^{1,2}$, by the rules of Malliavin calculus we have $\mathbb{E}[\xi_u + \int_t^T f(s, u, U_{u,s}^k, V_{u,s}^k)ds | \mathcal{F}_t] \in \mathbb{D}^{1,2}$ and hence $U_{u,t}^{k+1} \in \mathbb{D}^{1,2}$. Consequently for $\int_t^T V_{u,s}^{k+1} dW_s = \xi_u + \int_t^T f(s, u, U_{u,s}^k, V_{u,s}^k)ds - U_{u,t}^{k+1}$ again Lemma 5.1 in [6] yields $V_{u,t}^{k+1} \in \mathbb{D}^{1,2}$. Given these properties we have for $0 \leq v \leq u \leq t \leq T$

$$\begin{aligned} D_v U_{u,t}^{k+1} &= D_v \xi_u - \int_t^T D_v V_{u,s}^{k+1} dW_s \\ &\quad + \int_t^T \left[(D_v f)(s, u, U_{u,s}^k, V_{u,s}^k) + \left\langle (\nabla f)(s, u, U_{u,s}^k, V_{u,s}^k), (D_v U_{u,s}^k, D_v V_{u,s}^k) \right\rangle \right] ds. \end{aligned}$$

We continue by showing that the sequence $(D_v U_{u,t}^k, D_v V_{u,t}^k)$ converges and identify its limit as $(D_v U_{u,t}, D_v V_{u,t})$ which in addition is a solution of (36).

If we assume that Eq. (36) has a solution $(D_v U_u, D_v V_u)$ then the usual moment estimation techniques combined with the current assumptions produce

$$\sup_{0 \leq v, u \leq T} \left\{ \|D_v U_u\|_{\mathcal{S}^{2p}} + \|D_v V_u\|_{\mathcal{H}^{2p}} \right\} < \infty, \quad p \geq 1. \quad (37)$$

Fix $N \in \mathbb{N}$ to be chosen later, fix $0 \leq v \leq u \leq T$, set $\delta = T/N$ and define a partition $\tau_i = i\delta$ for $i \in \{1, \dots, N\}$. Then a priori estimates yield for $0 \leq i \leq N-1$

$$\begin{aligned} A_{u,v,i}^{k+1} &= \|D_v U_{u,i}^{k+1} - D_v U_{u,i}\|_{\mathcal{S}^2([\tau_i, \tau_{i+1}])}^2 + \|D_v V_{u,i}^{k+1} - D_v V_{u,i}\|_{\mathcal{H}^2([\tau_i, \tau_{i+1}])}^2 \\ &\leq C \left\{ \mathbb{E} \left[|D_v U_{u,\tau_{i+1}}^{k+1} - D_v U_{u,\tau_{i+1}}|^2 \right] + B_{u,v,i}^k + C_{u,v,i}^k \right\} \end{aligned} \quad (38)$$

with

$$\begin{aligned} B_{u,v,i}^k &= \| |D_v b| |U_u^k - U_u| + |D_v c| |V_u^k - V_u| \|_{\mathcal{H}^2([\tau_i, \tau_{i+1}])}^2, \\ C_{u,v,i}^k &= \mathbb{E} \left[\left(\int_{\tau_i}^{\tau_{i+1}} \left[|b_s| |D_v U_{u,s}^k - D_v U_{u,s}| + |c_s| |D_v V_{u,s}^k - D_v V_{u,s}| \right] ds \right)^2 \right]. \end{aligned}$$

Since both b and c are bounded, Jensen's inequality yields

$$C_{u,v,i}^k \leq C \delta A_{u,v,i}^k$$

and hence, an induction argument combined with (37), (38) and the assumptions provides

$$\sup_{0 \leq u, v \leq T} A_{u,v,i}^k < \infty, \quad \text{for all } k \geq 0. \quad (39)$$

To estimate $B_{u,v,i}^k$, note that according to (A3), $\sup_{0 \leq v \leq T} \{ \|D_v b\|_{\mathcal{H}^2} + \|D_v c\|_{\mathcal{H}^2} \} < \infty$ and that according to the first part of the proof $(U^k - U, V^k - V) \rightarrow 0$ in $\mathcal{S}^{2p} \times \mathcal{H}^{2p}$, $p \geq 1$. Now choose N large enough to guarantee $\alpha = C\delta < 1$. Therefore for any $\eta > 0$ one finds a $K^* \geq 0$, independent of u, v for which

$$A_{u,v,i}^{k+1} \leq C \mathbb{E} \left[|D_v U_{u,\tau_{i+1}}^{k+1} - D_v U_{u,\tau_{i+1}}|^2 \right] + \eta + \alpha A_{u,v,i}^k, \quad \text{for } k \geq K^*.$$

The equation $D_v U_{u,T} = D_v U_{u,T}^k$ allows us to write for $i = N-1$ and $k \geq K^*$

$$\sup_{0 \leq u, v \leq T} A_{u,v,N-1}^k \leq \eta + \alpha^{k-K^*} \sup_{0 \leq u, v \leq T} A_{u,v,N-1}^{K^*}.$$

As a consequence, (39) implies that $\sup_{0 \leq u, v \leq T} A_{u,v,N-1}^k \rightarrow 0$ as $k \rightarrow \infty$. One can expand the argument and show recursively that for all $0 \leq i \leq N-1$ one has $\sup_{0 \leq u, v \leq T} A_{u,v,i}^k \rightarrow 0$. Summing over i , one arrives at

$$\sup_{0 \leq u, v \leq T} \left\{ \|D_v U_u^{k+1} - D_v U_u^k\|_{\mathcal{S}^2([0,T])}^2 + \|D_v V_u^{k+1} - D_v V_u^k\|_{\mathcal{H}^2([0,T])}^2 \right\} \xrightarrow{k \rightarrow \infty} 0.$$

The conclusion is that (U_u, V_u) are indeed Malliavin differentiable and a version of its Malliavin derivatives is given by the limit of $(D_v U_u^k, D_v V_u^k)$.

The last statement of our theorem follows from Lemma 5.1 in [6]. We write our BSDE (35) for terminal time t , apply the Malliavin derivative operator, and obtain by the quoted lemma for $0 \leq v \leq u \leq t \leq T$

$$\begin{aligned} D_v U_{u,t} &= V_{u,t} - \int_v^t D_v V_{u,s} dW_s \\ &\quad + \int_v^t \left[(D_v f)(s, u, U_{u,s}, U_{u,s}) + \langle \nabla f(s, u, U_{u,s}, U_{u,s}), (D_v U_{u,s}, D_v U_{u,s}) \rangle \right] ds. \end{aligned}$$

Choosing $v = t$ leads to the desired representation. \square

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